Greek Angles from Babylonian Numbers

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Models of planetary motion as observed from Earth must account for two principal anomalies: (1) the nonuniform speed of the planet as it circles the zodiac, and (2) the correlation of the planet's position with the position of the Sun. In the context of the geometrical models used by the Greeks, the practical difficulty is to somehow isolate the motion of the epicycle center on the deferent, which is responsible for the first anomaly, from the motion of the planet on its epicycle, which is responsible for the second anomaly. This then allows determination of, first, the eccentricity e and longitude A of the apogee of the deferent and the positions $\bar{\lambda}_0$ of the epicycle center and $\bar{\alpha}_0$ of the planet on the epicycle at some time t_0, and second, the radius r of the epicycle.

One way to isolate the motion of the epicycle center for the outer planets is to determine the longitude and time of oppositions of the planet with the mean Sun. At the moment of such an opposition the radius of the epicycle is pointing directly at the Earth and so the size of the epicycle has no impact. Even so, the determination of the longitude and time of opposition is not easy. It requires both a sequence of timed longitude measurements of the planet both before and after opposition, while the planet's motion is retrograde, and an adequately calibrated solar theory to get the corresponding longitudes of the mean sun. The actual longitude and time of opposition is then a result of some sort of data reduction scheme. Using just three such oppositions and an elegant iterative geometrical analysis, Ptolemy shows in *Almagest* X–XI how to determine e, A, $\bar{\lambda}_0$, and $\bar{\alpha}_0$.[1] It turns out, however, that the parameter values determined using this method are extremely sensitive to small variations in the input data. It seems likely, in fact, that the values of the final model parameters found in the *Almagest* are the culmination of a chain of developments using simpler methods that may well have extended over several decades, if not centuries.[2] With these simpler methods one can use several more oppositions, perhaps a dozen or more, to determine the parameters and to a large extent avoid the extreme sensitivity that afflicts the *Almagest* method.[3] Either way, once the values of e, A, $\bar{\lambda}_0$, and $\bar{\alpha}_0$ are determined, a simple analysis of a single observation of the planet away from opposition yields a value for the epicycle radius r. Implicit in both analyses is a rough knowledge of the mean motions on the deferent and the epicycle, but these are easily determined using well-known period relations.

Given the extensive observational program required to determine the oppositions of each outer planet with the mean Sun, it would certainly not be surprising to suppose that if a shortcut was available for getting the empirical information necessary for determining the model parameters, some Greek astronomer(s) would use it, either for some sort of preliminary or exploratory analysis, or to confirm a conventional analysis of the type outlined above, or perhaps even in lieu of real observations. Thus, a Greek astronomer might have realized that, to the extent that they agree with observation, the predictions of mean oppositions by Babylonian models could serve as useful proxies for real empirical observations.

In addition, however, a Greek astronomer might also have realized how to extract from System A models for the outer planets the very nonuniform time variation of model angles that his kinematical models are designed to produce, namely the equation of center $q = \lambda - \overline{\lambda}$, which is the difference between the true and mean longitude of the epicycle center. In particular, in the geometrical models, at least as formulated in the *Almagest*, the time t is the independent variable and the equation of center is given as a function of the eccentricity e and the angle

$$\begin{aligned} \alpha &= \omega_L t - A \\ &= \overline{\lambda} - A \end{aligned}$$

Then the equation of center q for the eccentric model is

$$\tan q = \frac{-2e\sin\alpha}{R + 2e\cos\alpha}$$

and for the equant

$$\tan q = \frac{-2e\sin\alpha}{\sqrt{R^2 - e^2\sin^2\alpha} + e\cos\alpha}$$

In the above equations the normalization is such that the total eccentricity of the eccentric model is the same as the earth–equant distance in the equant model, namely $2e$.

In System A models the true longitude λ is the independent variable and the time and mean longitude are functions of λ, so it is convenient to invert the above relations and give q as a function of the angle $\gamma = \lambda - A$. Thus for the eccentric we have

$$\sin q = -\frac{2e}{R}\sin\gamma$$

and for the equant

$$\tan q = \frac{-2e\sin\gamma}{\sqrt{R^2 - e^2\sin^2\gamma} - e\cos\gamma}$$

These expressions for the equation of center are qualitatively similar to each other but differ in some details. For both models $q = 0$ at apogee and perigee, which are in each model 180° apart, and the extreme values of q are symmetric about the λ–axis, with values $\pm\sin^{-1}(2e/R)$ for the eccentric and $\pm 2\tan^{-1}(e/R)$ for the equant, the latter being smaller than the former (see Figure 1). The models differ, however, in that while for the eccentric the extreme values are 180° apart, for the equant the distance from maximum to minimum, and hence crossing apogee, is smaller than the distance from minimum to maximum, crossing perigee. Consequently, for the eccentric we have

$$\frac{dq}{d\lambda} = -2e/R \quad \text{at apogee}$$
$$= +2e/R \quad \text{at perigee}$$

but for the equant

$$\frac{dq}{d\lambda} = \frac{-2e/R}{1-e/R} \quad \text{at apogee}$$
$$= \frac{2e/R}{1+e/R} \quad \text{at perigee}$$

so that the derivative of q for the equant is larger in absolute magnitude at apogee than at perigee. We will see later that this property of the equant is reflected in most System A models for the outer planets, although whether anyone realized that is at present unclear.

*********Fig 1 goes here********

In any event, however, note that knowledge of where $q = 0$ when its extrema are equidistant from the λ–axis gives the apogee and perigee of the deferent, and both the values of $dq/d\lambda$ at $q = 0$ and the extreme values of q give estimates for e/R. Our task now is to see how a Greek astronomer could with relative ease determine the effective equation of center $q(\lambda)$ implied by a System A model, and hence estimate e and A by comparing the functions given above with his System A estimates.
Thus the plan of this paper is to

(a) assume that some Greek astronomers were trying to explain the zodiacal anomaly using geometrical models in a way that isolates the zodiacal anomaly from the effects of the solar anomaly
(b) assume that the Greek astronomers understood enough about how System A planetary models work to realize that they might be useful for such an explanation
(c) discuss whether it is even possible to use System A models to estimate the parameters of Greek geometrical models using methods that are plausible for Greek mathematicians.

Regarding assumption (a), Ptolemy, speaking of Hipparchus, writes (from Toomer, ref. 1):

> "For, *we may presume*, he [Hipparchus] thought that one must not only show that *each planet has a twofold anomaly*, or that each planet has retrograde arcs which are not constant, and are of such and such sizes (whereas the other astronomers had constructed there geometrical proofs on the basis of a single unvarying anomaly and retrograde arc); nor that these anomalies can in fact be represented either by means of eccentric circles or by circles eccentric with the ecliptic, and carrying epicycles, *or even combining both, the ecliptic anomaly being of such and such size, and the synodic anomaly of such and such* (for these representations have been employed by *almost all of those who have tried* to exhibit the uniform circular motion by means of the so-called 'Aeon-tables', but their attempts were faulty and at the same time lacked proofs: some of them did not

achieve their object at all, the others only to a limited extent); but, he reckoned that one who has reached such a pitch of accuracy and love of truth throughout the mathematical sciences will not be content to stop at the above point, *like the others who did not care*…" (italics added).

So although Ptolemy does not directly say that Hipparchus left the information above in his writings, he is at least willing to 'presume' what Hipparchus was thinking – that it was important to explain both anomalies with geometrical models, and that will many people had tried, none had yet succeeded. If Ptolemy was willing to presume that, then I suggest we should also be willing. This conclusion is further supported by the Hindu planetary models, which are generally considered to be pre-Ptolemaic and which definitely are geometrical attempts to explain both anomalies,[4] and which may in fact be among those Ptolemy, and as he presumed, Hipparchus, were alluding to above.

Assumption (b) is supported by the fact that Greek papyri give System A planetary model results using parameters adopted by Greek astronomers.[5] Such adaptations show clearly that at least some Greek astronomers were capable of understanding the structure of System A models well enough to modify them and bring them into conformance with Greek astronomy and Greek calendars.

As will be explained in detail in the following, given assumptions (a) and (b) the question posed in point (c) can be answered in the affirmative. In particular, however, while there is very tentative evidence discussed below from a papyrus that might be associated with Mars that someone might have gone down the path we discuss, that evidence is far from conclusive and might even be an illusion. Nevertheless, that does not affect the main result of this paper – that the proposed scenario might have plausibly happened. However unsatisfactory all of the uncertainty may seem, it is really no different than the same uncertainty we face in many other places in the history of ancient mathematical astronomy. We can see that some development was possible, and we can see that if some development happened then it would explain something we do reliably see, but we simply have no direct evidence to support the original premise that the said development did occur.

Unlike the Greek geometrical models, which are kinematical and based on explicitly continuous motion in space and time, the Babylonian models are based entirely on simple arithmetic and aim to explain only the location in space and time of a sequence of synodic events, which are appearances of a planet in a given relationship to a uniformly moving Sun, which a Greek astronomer would identify as the mean Sun. In order to clearly understand the underlying phenomena, let us begin by considering conjunctions of an outer planet, say Jupiter, with the Sun, and for now ignore the fact that such conjunctions are not in fact visible. Suppose that we see such a conjunction at a time t when the Sun and Jupiter both have longitude λ. Let us now suppose that the next conjunction occurs at some known longitude $\lambda + \Delta\lambda$. Our problem is to find the time this next conjunction occurs. Since the Sun moves much faster around the zodiac than Jupiter does, the time interval Δt between conjunctions is the time it takes the Sun to complete one full orbit from λ back to λ, which is of course one year of $y \simeq 365\tfrac{1}{4}^{d}$, plus the time it take the Sun to go the additional distance $\Delta\lambda$. If the Sun moves uniformly, or more realistically, if we simply *assume* the Sun moves uniformly, whether it does or not, then the Sun will complete the final $\Delta\lambda$ in time $\Delta\lambda/\omega_S$, where $\omega_S \simeq 360^\circ / 365\frac{1}{4}^d$, so in total

$$\Delta t = y + \frac{\Delta\lambda}{\omega_S}$$

and in fact this is exactly how the Babylonian System A models computed the time interval between synodic events for the outer planets, except that they had to allow for the fact that they used *tithis*, schematic days equal to $1/30^{\text{th}}$ of a lunar month, instead of days, and they approximated the Sun's speed as $1^{\circ/tithi}$. Note that this computation of the time interval remains the same if we use oppositions rather than conjunctions, and in fact if you use events at *any* constant elongation of the planet and the Sun. This line of analysis was first elaborated by van der Waerden, who called it his "Sun-distance principle".[6]

Stations and horizon phenomena such as acronychal rising and first and last visibility were considered and modeled by Babylonian astronomers. However, it is difficult accurately determine by observation both the position and time of these events, and it is only approximately true that the planet has a fixed relation to a uniformly moving Sun, or the real Sun, for that matter, in a sequence of such events. Whether or not the Babylonian astronomers understood the calculation in this way is, for our purposes, not important. What is important is that a Greek astronomer might have realized that the mathematical constraints and assumptions built into System A models, and in particular the assumption of a fixed relationship to a uniformly moving Sun, match very well with the actual properties of mean oppositions in geometrical models.

The following discussion will use algebraic notation for the convenience of the modern reader, but there is of course no implication that any ancient mathematician used such methods. What is important is that all of the operations needed are within reach of known Hellenistic mathematical methods.

Fundamental to System A are integral period relations.[7] Let *Y, L, A* (not to be confused with apogee) and *Z* be positive integers, with *A* and *Z* relatively prime, and suppose that in *Y* years a planet, and most important for our purposes, the epicycle center, completes *L* revolutions in the zodiac, during which *A* synodic events of the same kind are observed, and the location of the synodic events completes *Z* revolutions in the zodiac. For the outer planets

$$Y = L + A$$

and for the inner planets

$$Y = L$$

with $Z = L$ for Saturn, Jupiter and Mercury and $Z = L - A$ for Mars and Venus. Then the mean values of the synodic arcs – the changes in longitude of the synodic event and the planet (and its epicycle) between successive synodic events in units of revolutions – are

$$\begin{aligned}\overline{\Delta\lambda_Z} &= \frac{Z}{A} \\ \overline{\Delta\lambda_L} &= \frac{L}{A} \\ &= \frac{Z}{A} \quad \text{for Saturn and Jupiter} \\ &= \frac{Z}{A}\left(1 + \frac{A}{Z}\right) \quad \text{for Mars}\end{aligned}$$

and the number (integer plus fraction) of mean synodic arcs in one revolution is

$$P = \frac{1}{\overline{\Delta\lambda_Z}} = \frac{A}{Z}$$

The interval in years between successive mean synodic events is

$$\overline{\Delta t} = T_A = \frac{Y}{A}$$

and the mean intervals in years for the epicycle and the event to circle the zodiac are

$$T_L = \frac{Y}{L} \quad \text{and} \quad T_Z = \frac{Y}{Z}$$

With y days per year, the corresponding mean speeds of the epicycle, the event, and the Sun, in units of degrees per day, are

$$\begin{aligned}\omega_L &= \frac{360^\circ L}{yY} = \frac{\overline{\Delta\lambda_L}}{\overline{\Delta t}} \\ \omega_Z &= \frac{360^\circ Z}{yY} = \frac{\overline{\Delta\lambda_Z}}{\overline{\Delta t}} \\ \omega_S &= \frac{360^\circ}{y} = \frac{\overline{\Delta\lambda_S}}{\overline{\Delta t}}\end{aligned}$$

Note that on average Saturn and Jupiter advance $\overline{\Delta\lambda_Z} = \overline{\Delta\lambda_L}$ and the Sun advances $360^\circ + \overline{\Delta\lambda_Z}$ in time T_A, while Mars advances $360^\circ + \overline{\Delta\lambda_Z}$ and the Sun advances

$720^\circ + \overline{\Delta\lambda_Z}$ in time T_A. Finally, since Mars advances 360° in time T_L, it requires the time interval $T_A - T_L = T_A / (1 + A/Z)$ to advance its final $\overline{\Delta\lambda_Z}$.

The relations given above are all average values, and in reality the arcs and intervals between successive synodic events fluctuate about the mean, but the period relations tell us that if a synodic event is observed at some longitude λ_0, then after Y years and A successive events, and not before, the location of the synodic event will return to λ_0, having circled the zodiac Z times, and the epicycle will have circled the zodiac L times, so that the event will have landed on $A-1$ distinct other values of λ in the process. System A models this by distributing A λ's on a circle at intervals I_i, $i = 1\ldots A$, and successive synodic events progress in steps of Z intervals.[8] In principle, the A intervals I_i are of arbitrary size subject to the constraint

$$\sum_{i=1}^{A} I_i = 360^\circ$$

but, in reality, synodic arcs and intervals vary smoothly as a function of longitude, so for this reason and for computational efficiency the Babylonian astronomers assumed that the intervals are constant in zones of the ecliptic, with the number N of zones in all known systems being between two and six, and usually not of equal size. With N zones of size α_i and constant interval I_i, there are $N_i = \alpha_i/I_i$ intervals in the ith zone, and

$$\sum_{i=1}^{N} N_i = A$$

Usually, but not necessarily, the N_i intervals completely fill each zone and N_i is an integer. A synodic arc $w_i = ZI_i$ is associated with each zone, so α_i/w_i is the number (integer plus fraction) of such arcs in the ith zone. Since there are P arcs in 360°, the values of w_i are subject to the constraint

$$\sum_{i=1}^{N} \frac{\alpha_i}{w_i} = \sum_{i=1}^{N} \frac{N_i I_i}{Z I_i} = \frac{1}{Z}\sum_{i=1}^{N} N_i = \frac{A}{Z} = P$$

Furthermore, if an event lands in the ith zone at some λ, then the next event will be Z intervals farther, regardless of the size of the intervals. Thus if the next position is still in the ith zone, then its longitude is $\lambda' = \lambda + w_i = \lambda + ZI_i$, but if that position is in the (i+1)th zone its longitude will be $\lambda' = \lambda + mI_i + (Z-m)I_{i+1}$, where $\lambda + mI_i$ is the longitude of the beginning of the (i + 1)th zone, and similarly if the event lies in the (i+2)th zone (as actually happens for Mars and Mercury).

Since $A/Z = T_Z/T_A$, the relation $\sum_i \frac{\alpha_i}{w_i} = \frac{A}{Z}$ may be rewritten as

$$\sum_i \frac{\alpha_i}{\left(\frac{w_i}{T_A}\right)} = \sum_i \frac{\alpha_i}{v_i} = T_Z$$

For Saturn and Jupiter $Z = L$ and $T_Z = T_L$, so the epicycle has mean speed $v_i = \frac{w_i}{T_A} = \frac{w_i}{\overline{\Delta t}}$ in the ith sector, and the sum of the times spent in each sector, which are α_i / v_i, is the time $T_Z = T_L$ required for the epicycle to complete a single trip around the zodiac.

For Mars we can use instead

$$\sum_i \frac{\alpha_i}{w_i} = \frac{A}{Z} = \frac{A}{L} \cdot \frac{Z+A}{Z} = \frac{T_L}{T_A}\left(1 + \frac{A}{Z}\right)$$

or

$$\sum_i \frac{\alpha_i}{\frac{w_i}{T_A'}} = T_L$$

where $T_A' = T_A / (1 + A/Z) = T_A - T_L$.

There is one remaining problem to be addressed. If the planet is in opposition to the mean Sun, and in time interval $T_A = y + \overline{\Delta\lambda_Z} / \omega_S$ the planet advances an amount w_i while the mean Sun advances an amount $\overline{\Delta\lambda_Z}$, then at the end of the time interval the planet is not in fact in opposition to the mean Sun. System A planetary models resolve this problem by adjusting the time interval for a synodic arc $\Delta\lambda$ of length w_i according to

$$\begin{aligned}
\Delta t &= \overline{\Delta t} + \frac{y}{360^\circ}(\Delta\lambda - \overline{\Delta\lambda_Z}) \\
&= T_A + \frac{(w_i - \overline{\Delta\lambda_Z})}{\omega_S} \quad \text{for Saturn and Jupiter} \\
&= T_A' + \frac{(w_i - \overline{\Delta\lambda_Z})}{\omega_S} \quad \text{for Mars}
\end{aligned}$$

so that the time interval required for the planet to advance w_i differs from the mean time by the amount that the time required for the Sun to advance w_i differs from the time required to advance the mean distance $\overline{\Delta\lambda_Z}$.[9] Note that for Saturn and Jupiter, T_A is close to 400 days, while $(w_i - \overline{\Delta\lambda_Z}) / \omega_S$ is about ±1 day for Saturn and ±3 days for Jupiter, so the correction from mean to true time is not significant. However, for Mars T_A' is about

93 days and $(w_i - \overline{\Delta\lambda_Z}) / \omega_S$ varies between about -19^d and 41^d, so the correction is quite important.

The adjustment of the time intervals, and hence the implied speeds of the epicycle, does not upset the constraint on the w_i values. For Saturn and Jupiter the true speeds in each sector are now

$$\begin{aligned} v_i' &= \frac{w_i}{\Delta t} = \frac{w_i}{T_A + \dfrac{w_i - \overline{\Delta\lambda_Z}}{\omega_S}} \\ &= \frac{w_i}{T_A\left(1 + \dfrac{w_i}{\omega_S T_A} - \dfrac{\overline{\Delta\lambda_Z}}{\omega_S T_A}\right)} \\ &= \frac{v_i}{1 + \dfrac{v_i}{\omega_S} - \dfrac{\omega_Z}{\omega_S}} \end{aligned}$$

so we have

$$\begin{aligned} \sum_i \frac{\alpha_i}{v_i'} &= \sum_i \frac{\alpha_i}{v_i}\left(1 + \frac{v_i}{\omega_S} - \frac{\omega_Z}{\omega_S}\right) \\ &= \sum_i \frac{\alpha_i}{v_i} + \frac{1}{\omega_S}\sum_i \alpha_i - \frac{\omega_Z}{\omega_S}\sum_i \frac{\alpha_i}{v_i} \\ &= T_Z + \frac{360^\circ}{\omega_S} - \frac{\omega_Z T_Z}{\omega_S} \\ &= T_Z + y - y \\ &= T_L \end{aligned}$$

Repeating the above for Mars using $T_A' = T_A / (1 + A/Z) = T_A - T_L$ instead of T_A yields exactly the same result:

$$\sum_i \frac{\alpha_i}{v_i'} = T_L$$

Thus the constraint imposed by the period relation still holds exactly, and the epicycle in each case still completes one revolution in T_L years. In passing, note that if we had assumed that

$$\Delta t = T_A + p\left(w_i - \overline{\Delta\lambda_Z}\right)$$

for any value of p, then the periodicity will still be preserved.

Using the above analysis it is now straightforward to find the equation of center $q(\lambda) = \lambda - \bar{\lambda}$ for the outer planets. Since each System A model is a sequence of zones, we have that $\Delta\lambda_i = \alpha_i$ in each zone. Since v'_i is constant in each zone, the time required to traverse each zone is $\Delta t_i = \alpha_i / v'_i$ and during that time interval the mean longitude advances by the amount $\Delta\bar{\lambda}_i = \omega_L \Delta t_i$. Thus, from the beginning to the end of each zone, q_i changes by the amount

$$\begin{aligned} \Delta q_i &= \Delta\lambda_i - \Delta\bar{\lambda}_i \\ &= \alpha_i (1 - \frac{\omega_L}{v'_i}) \end{aligned}$$

and $dq/d\lambda$ in each zone is, for the piecewise linear System A models, exactly equal to

$$\frac{dq}{d\lambda} = \frac{\Delta q_i}{\alpha_i} = 1 - \frac{\omega_L}{v'_i}$$

From the values Δq_i one can reconstruct q and adjust its level so that the extreme values are equidistant from the λ–axis, and the longitude A of apogee will be where $q(\lambda)$ crosses the axis in a descending direction. Then, from the extreme values of q and from the slopes of q in the zones where q crosses the λ–axis, we can get estimates of e.

The parameters defining the various planetary models are available in many places.[10] The following charts compare for each outer planet the functions $q(\lambda)$ derived from the known System A models with the $q(\lambda)$ that follows from the equant using the parameters in the *Almagest*.[11]

*****Fig 2 goes here******

*****Fig 3 goes here******

*****Fig 4 goes here******

Although the discussion to this point has been entirely hypothetical, there is some evidence that might suggest the analysis discussed above was actually done by some ancient Greek astronomer. Neugebauer suggested that the contents of two Greco-Roman papyrus fragments are evidence for a greatly modified Babylonian System A scheme for Mars,[12] based on the fact that heading for one of the six tables is Taurus–Gemini, which is also one of the six System A zones for Mars and, according to Neugebauer, the only known use of a Taurus–Gemini pairing. Jones clarified and extended Neugebauer's analysis to reconstruct the six underlying values of w_i used in the scheme,[13] and suggested that the papyrus might be modeling a Greek kinematic model based on eccentricity $2e \simeq 5$, rather than the conventional $2e = 12$. However, when a single table of very similar but not identical structure turned up in the papyri of Oxyrhynchus, the association with Mars and System A seemed to Jones less probable.[14] Nevertheless, in light of the above analysis in this paper, and in spite of the tenuous connection of the papyrus with either Mars or System A, it is worth considering an alternative interpretation, namely that the papyrus documents an analysis of Mars' equation of center q from a System A scheme, much as outlined above. Indeed, let us consider a Mars scheme as above

$$\sum_i \frac{\alpha_i}{\dfrac{w_i}{T'_A\left(1+\dfrac{w_i}{T'_A}-\dfrac{\overline{\Delta\lambda}_Z}{T'_A}\right)}} = \sum_i \frac{\alpha_i}{\left(\dfrac{w'_i}{T'_A}\right)} = T_L$$

and assume the numbers given in the papyrus are not w_i values, as assumed by Neugebauer and Jones, but are instead w'_i values. Then from the w'_i values one may derive both the underlying w_i values for the scheme, which are

$$w_i = \frac{w'_i(1-\overline{\Delta\lambda}_Z / T'_A)}{(1-w'_i / T'_A)}$$

as well as the corresponding equation of center $q(\lambda)$, which is shown in the Mars chart. If this interpretation is correct, then the scheme in the papyrus is clearly not derived directly from the standard Babylonian System A scheme, but we cannot rule out that it was derived from some other scheme that does not differ too much from the standard scheme. Indeed, it seems possible that it might be an incomplete account of an attempt to derive a System A scheme starting with a geometric model of Mars, but I see no way to advance the argument further given that the papyrus gives no hint about how the numbers were ever used.

The apogees and perigees in the various System A models are shown in Table 1. Note that in some case the apogees and perigees are not 180° apart.

*******Table 1 goes here*********

To get the e implied by a System A model, we compute $\dfrac{dq}{d\lambda} = 1 - \dfrac{\omega_L}{v'_i}$ for the zones which ascend and descend through the λ– axis, and from these slopes we compute the effective e from the System A values of $dq/d\lambda$ using the equations given earlier. The results are shown in Table 2.

*******Table 2 goes here*********

For comparison, the *Almagest* values of e give the slopes in Table 3.

*******Table 3 goes here*********

Generally, one would expect the widest zones to be associated with the fastest speeds, and vice versa, and so the perigee would fall in the widest zone and the apogee in the shortest. This is indeed the case for Jupiter, and for Mars all of the zones are equal width (60°) so the rule is indeterminate. For Saturn, the usual two-zone System A model has the situation reversed – the slow zone is shorter than the fast zone. However, there is an alternate Saturn model, denoted A′ in the table, in which the slow zone is wider, in agreement with expectations.[15]

The three Jupiter System A models have the same minimum and maximum synodic arcs (30° and 36°), and so give the same values for $dq/d\lambda$ and e(A) and e(P). Note that e(A) agrees exactly with the *Almagest* value, but e(P) is too small. For the Babylonian Mars model both e(P) and e(A) are too large, and for the Mars model suggested by the papyrus, both e(P) and e(A) are too small. For Saturn, all of the values are too small.

Note also that for all the Mars and Jupiter models, and Saturn System A′ but not System A, the values of $dq/d\lambda$ are generally of the form suggested by the equant: $dq/d\lambda$(A) is more negative than $dq/d\lambda$(P) is positive. Whether this reflects some observable property of the synodic arcs, or whether it was noticed by our assumed Greek astronomer and somehow motivated the equant, or is simply an artifact of the System A construction constraints, is not clear.

An alternative method to find e is to use the extreme values of $q(\lambda)$. For the eccentric model these are $\sin^{-1}(2e/R)$, and for the equant $2\tan^{-1}(e/R)$. The resulting values are in Table 4.

*******Table 4 goes here*********

In the models with only two zones the sharp breaks at the maximum and minimum values of q do not model the function very well and are responsible for the discordant values of e derived from the extreme values of q. The models with more zones generally improve the models near the maxima and this is reflected in better estimates of e.

We have seen that for oppositions of the outer planets with the mean Sun it is useful to think of System A in terms of a discrete set of A points distributed around the zodiac, with subsets of the points evenly distributed at intervals I_i in zones of width α_i. Within the ith zone it is useful to think of the epicycle center as moving through adjacent points at a constant speed

$$v_i' = \frac{w_i}{\Delta t} = \frac{w_i}{T_A + \dfrac{w_i - \overline{\Delta\lambda}_Z}{\omega_S}}$$

so in time interval $\Delta t_i = I_i / v_i'$ the epicycle center will advance in mean longitude by an amount

$$\Delta L_i = \omega_L \Delta t_i$$

where ω_L is the mean motion in longitude of the epicycle center. Then the change in the equation of center is simply $\Delta q_i = I_i - \Delta L_i$. Note that the additive correction to T_A in the expression for v_i' accounts for the fact that when the epicycle center advances by a distance w_i instead of the average distance $\bar{w} = \overline{\Delta\lambda}_Z$, the faster–moving mean Sun will return to opposition with the epicycle center in time $\Delta t = y + \frac{w_i}{\omega_S}$ instead of the average time $\overline{\Delta t} = T_A = y + \frac{\bar{w}}{\omega_S}$ (in these expressions we have y for Jupiter and Saturn, which becomes $2y$ for Mars).

Note also that instead of referring the opposition to the mean Sun and the epicycle center, we could just as well refer to the planet and its epicycle center, as long as we use the mean motion on the epicycle referred not to the epicycle apogee, in which case the speed would be the mean motion in anomaly $\omega_A = \omega_S - \omega_L$, but to a fixed sidereal direction, in which case the speed is simply $\omega_S = \omega_L + \omega_A$. Relative to such a fixed sidereal direction, e.g. the apogee of the deferent, the epicycle center advances on the deferent by w_i and the planet advances on its epicycle by 360° + w_i in time interval

$$\Delta t = y + \frac{w_i}{\omega_S} \qquad \text{for Jupiter and Saturn}$$

$$\Delta t = 2y + \frac{w_i}{\omega_S} \qquad \text{for Mars}$$

Thus, we can understand that the reason that a geometrical analysis of the System A models actually works, and that the same System A model works for multiple synodic phenomena, is that the various synodic phenomena for the outer planets are, in fact, all reasonably consistent with the System A assumption that synodic intervals in time are linear functions of the synodic arcs in longitude, and that the slope of the line is $1/\omega_S$, where ω_S is the uniform speed of the planets on its epicycle relative to a fixed sidereal direction.

Inferior and superior conjunctions of the inner planets with the true Sun are, in principle, essentially identical to the above treatment of the outer planets. For an inner planet the epicycle center moves with variable speed around the deferent, the variation is about the mean speed $\omega_L = \omega_S$ of the Sun, and the planet moves uniformly on its epicycle with speed $\omega_P = \omega_S + \omega_A$ measured with respect to a fixed sidereal direction.[16] Thus all of the results discussed above would be immediately applicable if not for the fact that (1) conjunctions of an inner planet with the Sun are not observable due to the glare of the Sun, and (2) for a variety of reasons, but principally the motion in latitude, the synodic time intervals and synodic arcs in longitude of the first and last visibilities do not, in general, all have the same linear relationship and so do not satisfy the expected relationships

$$\Delta t = \frac{360^\circ + w_i}{\omega_P} \qquad \text{for Mercury}$$

$$\Delta t = \frac{720^\circ + w_i}{\omega_P} \qquad \text{for Venus}$$

As a result, while the System A models for Mercury and Venus do a credible job of modeling the observed synodic phenomena,[17] they do not do a good job of modeling the nonuniform movement of the epicycle center around the deferent and so do not yield useful information regarding the parameters of geometrical models.

Synodic months for the Moon–Sun system are strongly influenced by the nonuniform motion of both the Moon and the Sun. The figure below shows the synodic time interval Δt versus the synodic arc $\Delta\lambda$ for all full moons in the 100 year interval beginning with –200 January 1.

*****Fig 5 goes here******

The events along the left and right vertical edges occur when the Sun is near apogee and perigee, respectively, and similarly, those along the lower and upper horizontal edges occur when the Moon is near perigee and apogee. If in reality the Sun was moving nonuniformly and the Moon uniformly, the distribution would collapse vertically and the events would all be on a line (purple in the figure above) of relatively shallow slope of the form

$$\Delta t \simeq 27\tfrac{1}{3}^d + \frac{w}{\omega_L}$$

where ω_L is the lunar mean motion in longitude. Syzygies at the extreme left and right ends of this line therefore have the Sun moving at minimum and maximum speed, respectively, and the Moon moving at mean speed. On the other hand, if in reality the Moon was moving nonuniformly and the Sun uniformly, the distribution would collapse horizontally and the events would all be on a line (red in the figure above) of relatively steep slope of the form

$$\Delta t \simeq \frac{w}{\omega_S}$$

Syzygies at the extreme upper and lower ends of this line therefore have the Moon moving at minimum and maximum speed, respectively, and the Sun moving at mean speed. Note that the lines cross at the point $w = 29.11°$ and $\Delta t = 29.53^d$, which are the mean synodic arc and synodic time for the Moon–Sun system.

The System A model for the Moon incorporates the nonuniform motion of both the Moon and the Sun, and so the analysis is somewhat more involved that we found for the outer and inner planets.[18] The synodic event is assumed to advance by arcs of w_1 = 30° and w_2 = $28\frac{1}{8}$° in its two zones and the time interval Δt is 29^d plus two corrections, both given in units of large hours *H,* which are four regular hours or 1/6th of a day (the major unit in the vertical scale of the figure). One correction, column G, depends on how far the Moon is from its apogee, and the second, column J, depends on how far the Sun is from its apogee. Column G assumes that the Sun is in its fast zone, and so near the right vertical edge of the chart, and the correction to Δt given by column G varies between about 2 $2/3^H$ to just short of 5^H, in good agreement with the real variation, as shown in the figure below.[19] When the Sun is in its fast zone, and so near the right edge of the chart, column J gives no correction, but when the Sun is in its slow zone, and so near the left vertical edge of the chart, column J subtracts about $0;57^H$ from Δt, again in good agreement with reality.

*****Fig 6 goes here******

In order to estimate the values of *e/R* and *r/R* implied by the System A lunar theory, we
may use the values of $\Delta\lambda$ and Δt at the endpoints of the red and green lines. In general, syzygies on the red line have

$$\lambda_S(t) = \overline{\lambda}_S(t) + q_S(t) = \overline{\lambda}_M(t)$$

and from any two successive syzygies at times t_1 and t_2 we get the relation

$$\Delta\lambda = \omega_S \Delta t + q(t_2) - q(t_1) = \omega_M \Delta t$$

where ω_S and ω_M are the mean motions in longitude of the Sun and Moon. Syzygies at the ends of the red line have $\alpha_S = \lambda_S - \overline{\lambda}_S = 0°$ or 180° and so $q(t_1) = 0°$ and since in the time interval Δt we have α_S advancing by the amount $\omega_S\Delta t$ we have, with $\eta = \omega_M - \omega_S$,

$$q(t_2) = \sin^{-1}\left(\frac{e}{R}\sin\left(\Delta\lambda\right)\right) = \eta\Delta t$$

at apogee and

$$q(t_2) = \sin^{-1}\left(\frac{e}{R}\sin\left(\Delta\lambda + 180°\right)\right) = \eta\Delta t$$

at perigee. Solving these for *e/R* we find *e* = 2.26 and *e* = 1.92, which average to *e* = 2.09 and which is close to the true value.

Similarly, for syzygies on the green line we have

$$\lambda_M(t) = \overline{\lambda}_M(t) + q_M(t) = \overline{\lambda}_S(t)$$

and from any two successive syzygies at times t_1 and t_2 we get the relation

$$\Delta\lambda = \omega_M \Delta t + q(t_2) - q(t_1) = \omega_S \Delta t$$

In this case α_M advances by the amount $\Delta\lambda - (\omega_M - \omega_a)\Delta t$ where ω_a is the lunar mean motion in anomaly, and we find

$$\frac{r}{R} = \frac{\sin(\eta\Delta t)}{\sin(\Delta\lambda - (\omega_M - \omega_a)\Delta t)}$$

at apogee and

$$\frac{r}{R} = \frac{\sin(\eta\Delta t)}{\sin(\Delta\lambda - (\omega_M - \omega_a)\Delta t + 180°)}$$

at perigee, which yield $r = 4.74$ and $r = 6.39$,[20] and which average to 5.57, somewhat larger than the true value which is near 5.25.

In summary, it seems plausible that a Greek astronomer with a reasonable understanding of Babylonian System A models for the outer planets and the Sun–Moon could have used those models to estimate the zodiacal variation of the equation of center for each planet, and from this, approximate values for the eccentricity *e* and longitude of apogee *A* required for geometrical models. The same method would work for the inner planets if conjunctions were observable, but they are not, and the variation of the observable synodic events – first and last morning and evening visibilities – is dominated more by the motion of the planet in latitude than the nonuniform motion of the epicycle center.

Table 1. The deferent apogee and perigee values deduced from the various System A models and compared to the *Almagest* values.

	apogee	perigee	*Almagest*
Mars A	126°	306°	115.5°
Mars (papyrus)	115.5(!)	297.5	
Jupiter A	165.5	342.5	161
Jupiter A′	154.5	345.5	
Jupiter A″	166	340	
Saturn A	230	50	233
Saturn A′	225	45	

Table 2. The eccentricities of the deferent deduced from the slopes *dq/dλ* at apogee and perigee of the various System A models and compared to the *Almagest* values.

	dq/dλ(A)	*dq/dλ*(P)	equant		eccentric		*Almagest e*
			e(A)	*e*(P)	*e*(A)	*e*(P)	
Mars A	–0.2970	+0.2183	7.76	7.35	8.91	6.55	6.00
Mars (papyrus)	–0.1988	0.1476	5.43	4.78	5.96	4.42	6.00
Jupiter A, A′, A″	–0.0961	+0.0727	2.75	2.26	2.88	2.18	2.75
Saturn A	–0.0775	+0.0967	2.23	3.05	2.32	2.90	3.42
Saturn A′	–0.0932	+0.0834	2.67	2.61	2.80	2.50	3.42

Table 3. The slopes *dq/dλ* at apogee and perigee that follow from the *Almagest* equations of center for the outer planets.

	Value of *e*	*2e/R*	*dq/dλ*(A)	*dq/dλ*(P)
Mars	6	0.2000	–0.2222	+0.1818
Jupiter	2.75	0.0917	–0.0961	+0.0876
Saturn	3.4167	0.1139	–0.1208	+0.1078

Table 4. The eccentricities of the deferent deduced from the extreme values of the various System A models and compared to the *Almagest* values.

	$(q_{max} - q_{min})/2$	e(eccentric)	e(equant)	*Almagest e*
Mars A	13.21	6.85	7.04	6.00
Mars (papyrus)	10.26	5.34	5.43	6.00
Jupiter A	7.45	3.88	3.92	2.75
Jupiter A′	5.77	3.01	3.03	2.75
Jupiter A″	4.81	2.52	2.52	2.75
Saturn A	7.73	4.04	4.07	3.42
Saturn A′	7.92	4.14	4.18	3.42

Figure 1. A comparison of the equation of center for an eccentric (dashed line) and an equant (solid line). The eccentricity is intentionally large so that the differences can be seen. However, the differences are much smaller for all real planets relevant for ancient astronomy.

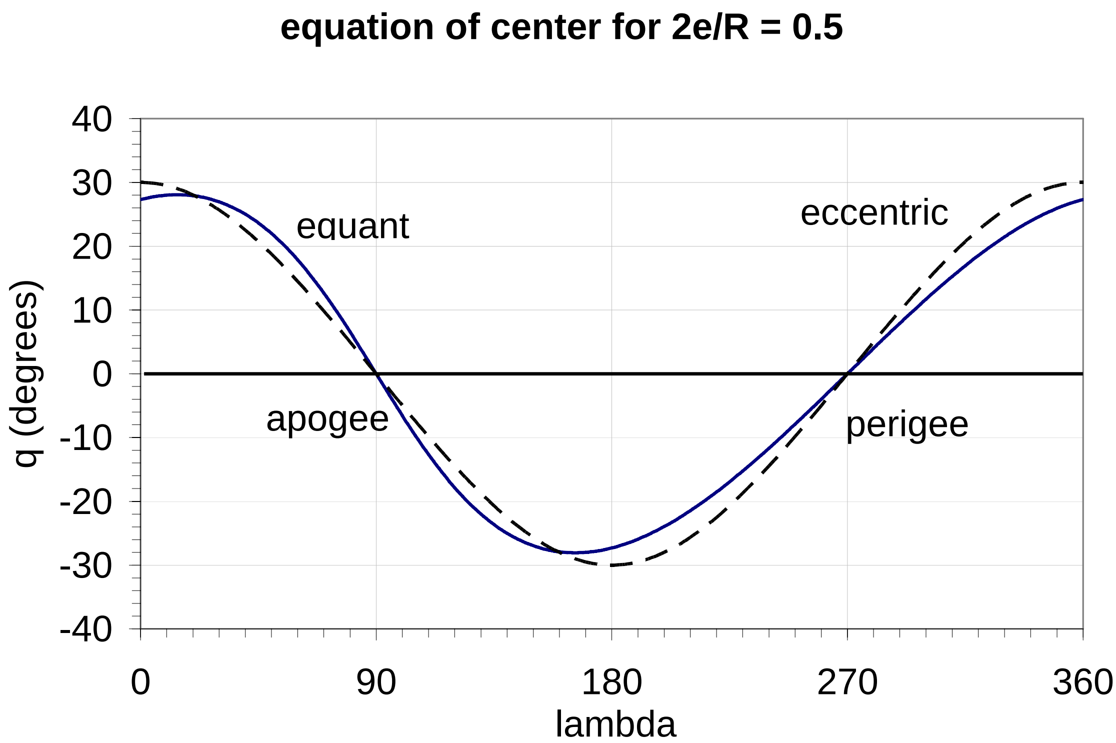


Figure 2. The equation of center for Mars from the *Almagest* (dotted line), the Babylonian System A (solid), and a Greek papyrus (ref. 13, dashed line).

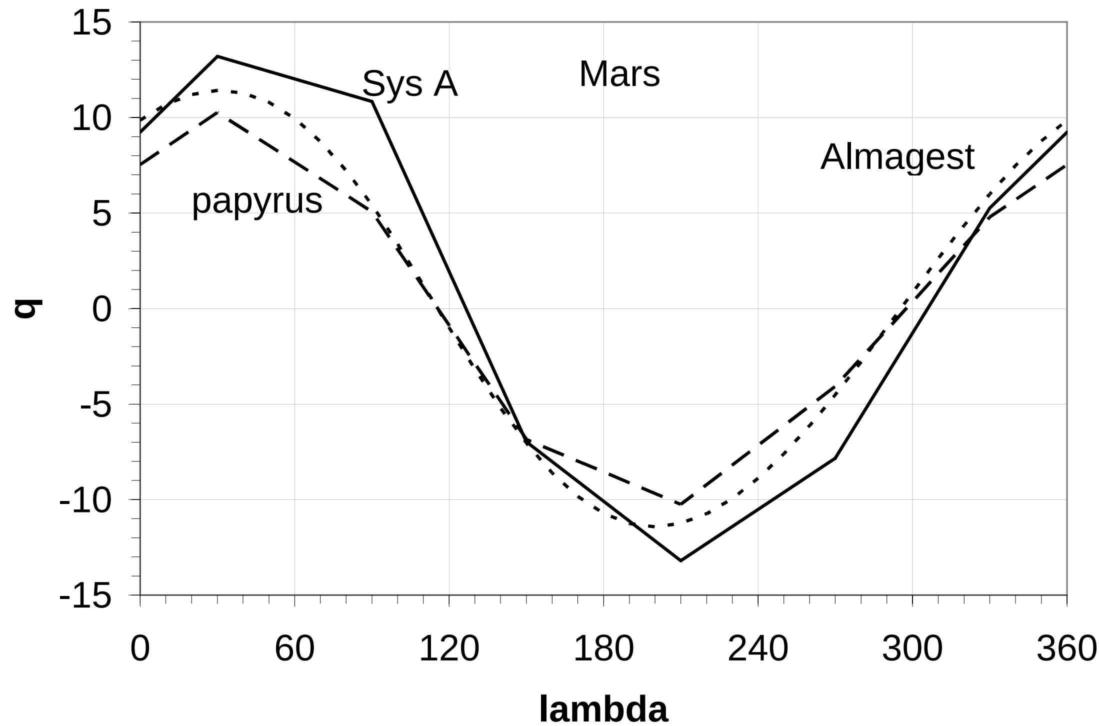


Figure 3. The equation of center for Jupiter from the *Almagest* (dotted line), the Babylonian System A (solid) and System A′ (dashed), and a Greek papyrus (ref. 5, dot-dashed line).

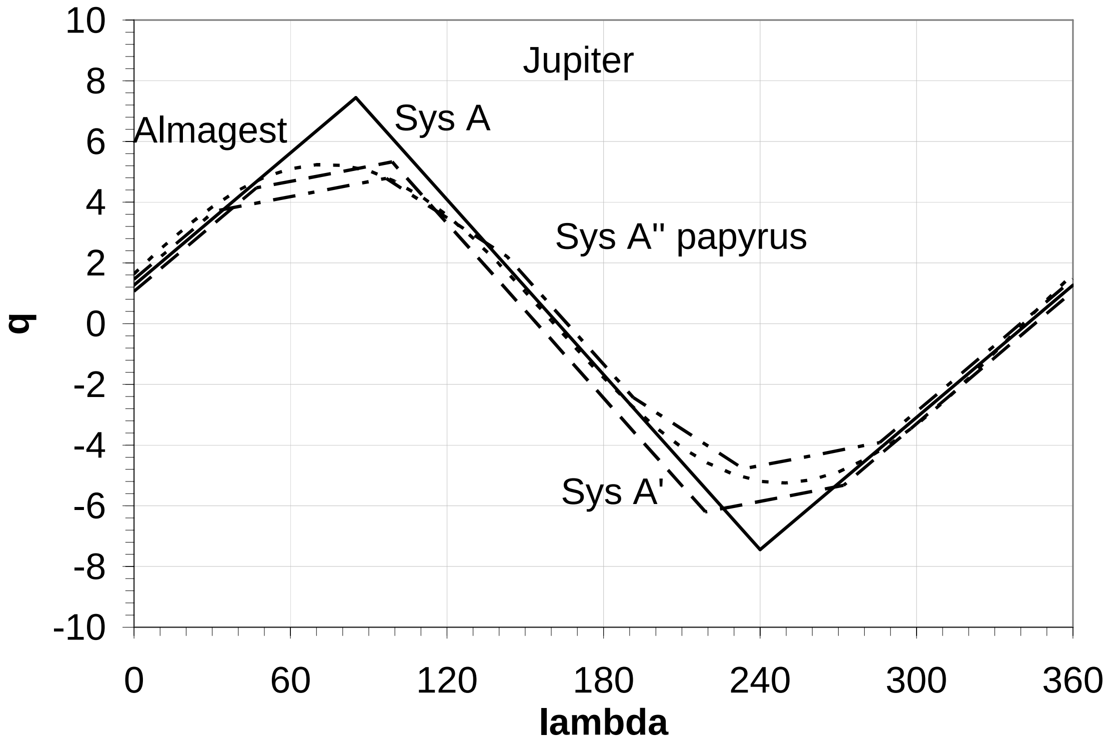


Figure 4. The equation of center for Saturn from the *Almagest* (dotted line) and the Babylonian System A (solid) and System A′ (ref. 10, dashed),

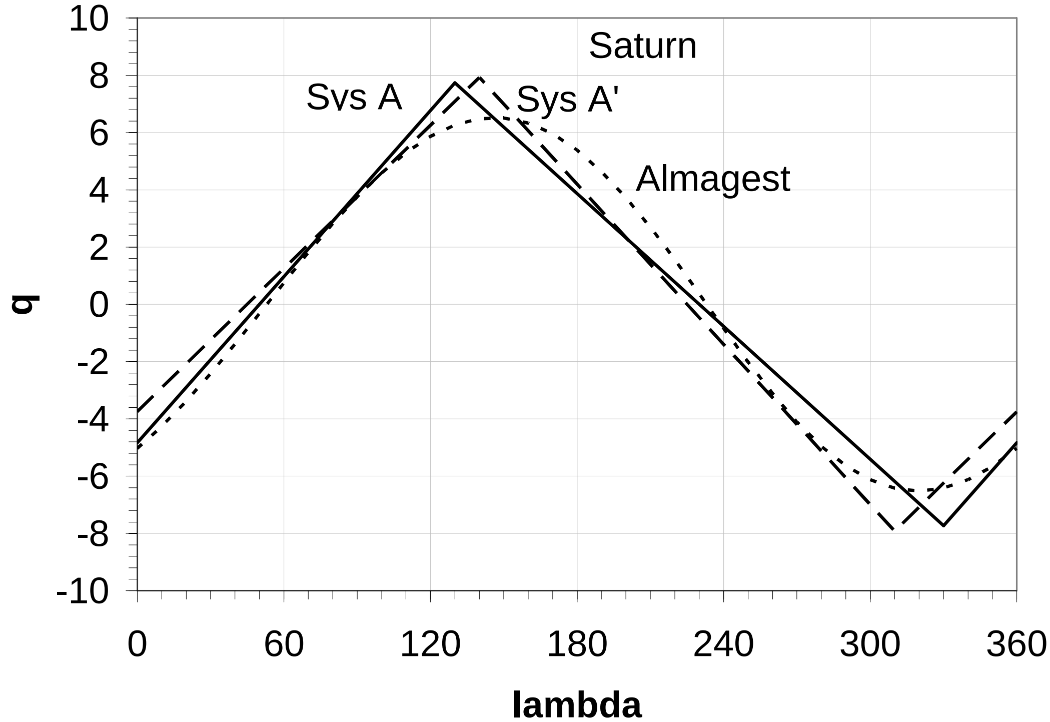


Figure 5. The distribution of synodic arcs ($d\lambda$) vs. synodic time intervals (*dt*) for all full moons between –200 BCE and –100 BCE (small triangles). The nearly horizontal line through the center shows the same distribution for oppositions of the real Sun and the mean Moon. The nearly vertical line through the center shows the same distribution for oppositions of the mean Sun and the real Moon. In all cases the points anywhere along the upper and lower edges have the Moon near apogee and perigee, respectively, and the points anywhere along the left and right edges have the Sun near apogee and perigee.

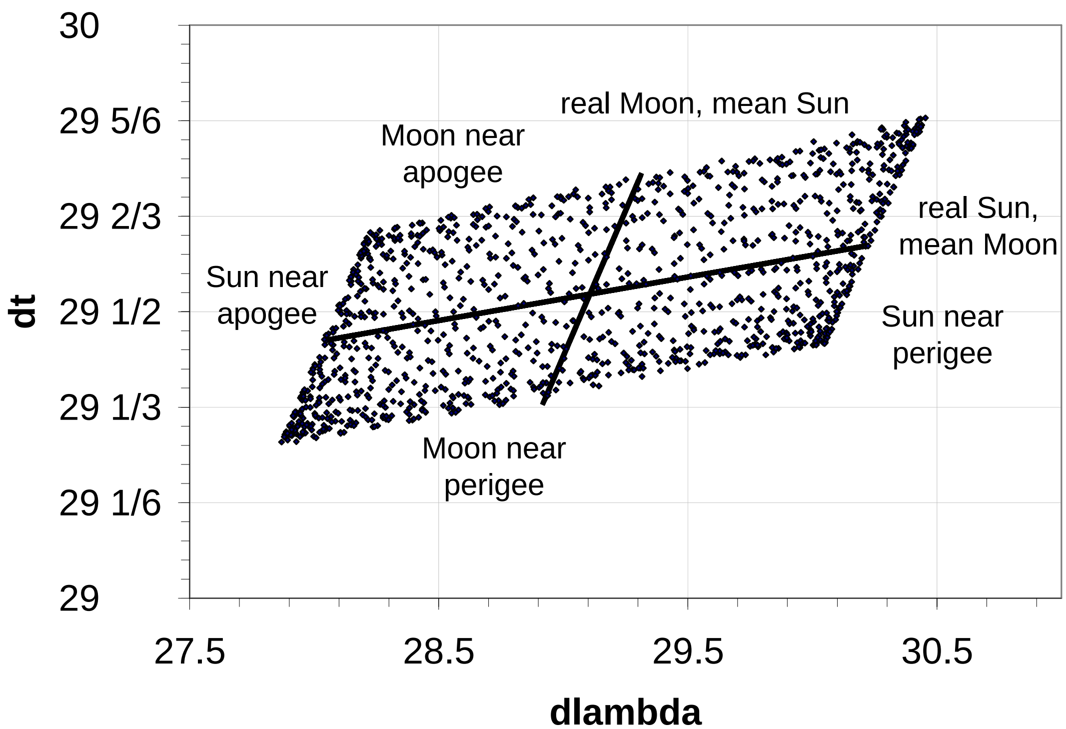
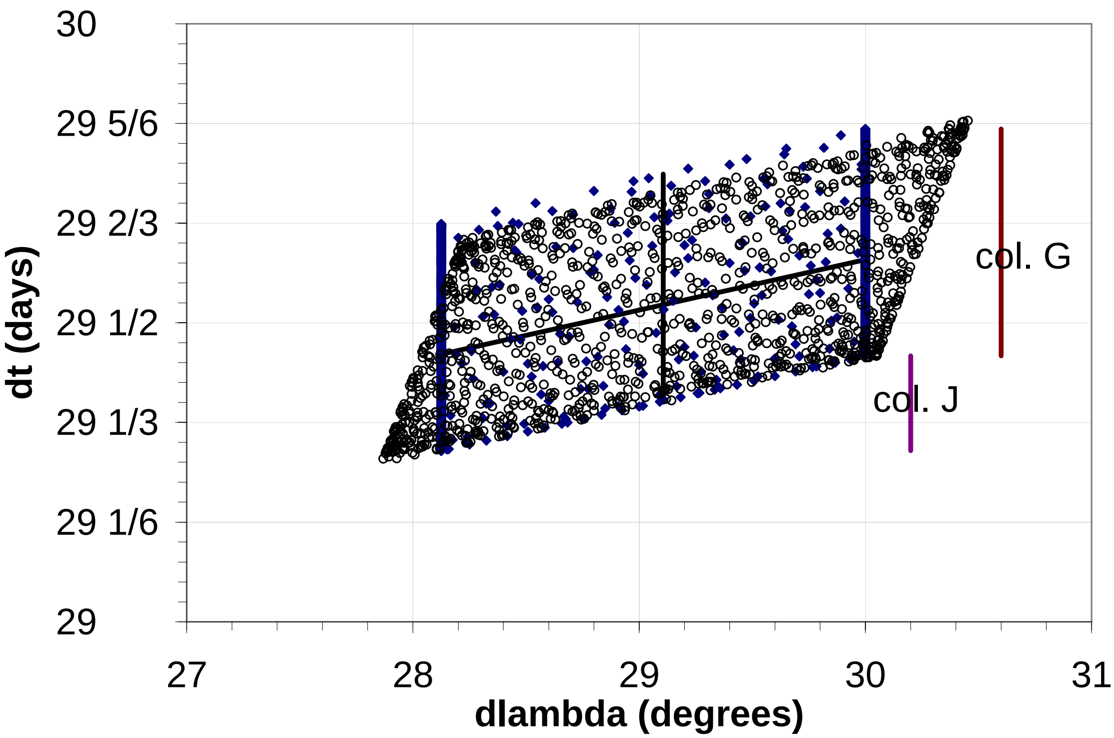


Figure 6. The distribution of synodic arcs ($d\lambda$) vs. synodic time intervals (*dt*) from the Babylonian lunar System A for all full moons between –200 BCE and –100 BCE (small triangles) compared to reality (open circles). The large vertical extension in time shows the effect of column G and is entirely due to lunar anomaly. The small vertical extension in time shows the effect of column J and is entirely due to solar anomaly. The solid lines through the center thus show the distribution of System A full moons if there was only a lunar anomaly (vertical line) or only a solar anomaly (nearly horizontal line). The heavy accumulation of points along the left and right edges for System A reflects the fact that lunar System A has only two zones, and most synodic arcs are entirely contained in one of the zones. The sparse points not on an edge are the arcs that cross a zone boundary. The lack of a slant to the right in the System A distributions is due to the fact that System A adjusts the synodic time interval, but not the synodic arc, for the effects of lunar (col. G) and solar (col. J) anomaly.



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[10] Alexander Jones, "Studies in the Astronomy of the Roman Period III. Planetary Epoch Tables", *Centaurus* **40** (1998) 1–41, see particularly Table 2, p. 6. For the Jupiter System A″ model see *ibid*. (ref. 5). For the Saturn System A′ model see Noel M. Swerdlow, *The Babylonian Theory of the Planets*, (Princeton, 1998), 93–94.
[11] In particular, the charts use the tropical longitude of apogee values that Ptolemy assigns for about A.D. 140. The correspondence between these tropical values and the sidereal Babylonian values is approximately $\lambda_s = \lambda_t + 3.08^\circ + 0.013825^{\circ/y}(y_s - y_t)$, where λ_s is a longitude in year y_s, roughly 200 B.C. for the Babylonian System A models, and λ_t is a longitude in year y_t. See, for example, F. Rochberg, *Babylonian Horoscopes* (Philadelphia, 1998) 19–20.
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[16] Measuring the mean motion of a planet, outer or inner, on its epicycle with respect to a fixed sidereal direction is seen in the Keskintos Inscription (*ca*. 100 B.C.): Alexander Jones, "The Keskintos Astronomical Inscription: Text and Interpretations", *SCIAMVS* 7 (2006) 3-41; throughout all of ancient Hindu astronomy, generally thought to be of Greek origin and pre-Ptolemaic: Dennis W. Duke, "Mean Motions and Longitudes in Indian Astronomy, *Archive for History of Exact Sciences*, 62 (2008) 489-509, and references therein; and in Ptolemy's *Planetary Hypotheses*, but not the *Almagest*: Dennis Duke, "Mean Motions in Ptolemy's Planetary *Hypotheses*", *Archive for History of Exact Sciences*, 63 (2009) 635-654.

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[20] These two estimates of r from the System A lunar model are intriguingly similar to the values $r = 4.75$ and $r = 6.25$ that Ptolemy attributes to Hipparchus in *Almagest* IV 11.