Question 9: PDEs Spring 2008

Given the function $f(x, y)$, consider the problem:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{for } 0 < x < 1 \text{ and } 0 < x < 1$$

$$u(x,0) = u(x,1) = 0 \qquad \text{for } 0 \le x \le 1$$

$$u(0,y) = u(1,y) = 0 \qquad \text{for } 0 \le y \le 1.$$

a. Discuss how you would determine an approximate solution of this problem using a piecewise linear finite element method.

b. Discus the factors that affect the accuracy of finite element methods for the approximation solution of this problem.

8. *Numerical PDEs*

Given the functions $f(x,y)$ and $g(x,y)$, consider the problem:

$$
\begin{aligned}
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + g(x,y)u = f(x,y) \qquad & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\
u(x,0) = u(x,1) = 0 \qquad & \text{for } 0 \le x \le 1 \\
u(0,y) = u(1,y) = 0 \qquad & \text{for } 0 \le y \le 1.
\end{aligned}
$$

(a.) Discuss how you would determine an approximate solution of this problem using a piecewise linear finite element method.

(b.) Discuss the factors that affect the accuracy of finite element methods for the approximation solution of this problem.

11. Consider the two point boundary value problem (BVP)

$$-\frac{d^2u}{dx^2} + p\frac{du}{dx} + qu = f(x) \quad a < x < b$$

$$u(a) = 0 \qquad \alpha u(b) + u'(b) = 1$$

where p, q, α are scalars.

a. Write down a weak formulation of this problem. Show that a solution to this classical two point BVP is also a solution of your weak problem. Is the converse always true? Why or why not?

b. Suppose we want to approximate the solution of the weak problem using continuous, piecewise linear polynomials defined over a uniform partition x_j, $j = 0, \ldots, n+1$ of $[a, b]$ where $x_0 = a$, $x_{n+1} = b$. Write a discrete weak problem.

c. Assume that we use the standard "hat " basis functions. Show that once the basis functions are chosen we can write the discrete weak problem as a linear system. If $p = q = \alpha = 0$ what are the properties of this linear system? Explicitly determine the entries of the coefficient matrix when $p = q = \alpha = 0$ in this linear system assuming we use the midpoint rule to compute the entries.

d. Discuss the rates of convergence in both the H^1 and L^2 norms that you expect using continuous, piecewise linear polynomials.

6. *Partial differential equations: finite element method*

Consider the diffusion equation

$$u_t = \alpha u_{xx}$$

with the initial and boundary conditions

$$u(x, 0) = g(x), \quad u(0, t) = u_L, \quad u(1, t) = u_R.$$

The function $g(x)$ is prescribed over the interval $0 < x < 1$, and α, u_L and u_R are constants and $\alpha > 0$.

(a) The backward-time difference scheme can be used to convert the above initial-boundary value problem into a two-point boundary value problem (BVP) at every time step. Carry out the details of this step and develop this BVP. (20%)

(b) Develop a complete piecewise-linear Galerkin-type finite element scheme to solve the resulting boundary value problem derived in part (a). (70%)

(c) Comment on the numerical stability of the backward-time finite element scheme developed in (a) and (b) above (10%)

4. *Partial Differential Equations*

Consider the simple 1-D diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l, \quad t > 0$$

subject to

$$u(0,t) = u(l,t) = 0 \qquad t > 0 \qquad (b.c.)$$

$$u(x,0) = u_0(x), \qquad 0 < x \leq l \qquad (i.c.)$$

Using Galerkin approximation with piecewise linear basis function which satisfy the boundary conditions

$$\Phi_i(0) = \Phi_i(l) = 0$$

a. Show that if we use approximate solution

$$U(x,t) = \sum_{i=1}^{N} u_i(t)\Phi_i(x)$$

That we obtain

$$\sum_{j=1}^{N} \left\{ \frac{du_j}{dt} b_{ij} + u_j g_{ij} \right\} = 0 \qquad i = 1, 2, .., N$$

where

$$g_{ij} = \int_0^l \frac{d\Phi_i}{dx}\frac{d\Phi_j}{dx}\, dx \qquad i,j = 1,2,...,N(\text{stiffness matrix})$$

$$b_{ij} = \int_0^l \Phi_i \Phi_j \, dx(\text{mass matrix})$$

i.e., a system

$$B\dot{U} + GU = b \qquad (\text{here } b = 0)$$

where

$$B = \{(\Phi_j, \Phi_i)\}$$

$$G = \{a(\Phi_j, \Phi_i)\}$$

b. Show that if h is the element size, we have that matrices B and G have the following entries

$$B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix}$$

$$G = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

for piecewise linear hat basis functions on regular mesh.

(*Hint*: Use Sylvester's formula for integration)

2. *Partial Differential Equations* (Dr. Peterson)

a. Let Ω be a bounded domain in R^2 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \cap \Gamma_2 = \emptyset$. Consider the following PDE and boundary conditions for $u(x, y)$

$$-\Delta u + uu_x = f(x, y) \quad (x, y) \in \Omega$$

$$u = 0 \quad \text{on } \Gamma_1 \qquad \frac{\partial u}{\partial n} = 4 \quad \text{on } \Gamma_2$$

and the weak formulation

Seek $u \in \hat{H}^1$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uu_x v = \int_\Omega fv + 4\int_{\Gamma_2} v \quad \forall v \in \hat{H}^1$$

where $\hat{H}^1$ is all functions that are zero on Γ_1 and which possess one weak derivative. Here $\Delta u = u_{xx} + u_{yy}$ and $\partial u/\partial n$ denotes the derivative of u in the direction of the unit outer normal, i.e., $\nabla u \cdot \vec{n}$. Show that if u satisfies the classical boundary value problem then it satisfies the weak problem. Then show that if u is a sufficiently smooth solution to the weak problem, then it satisfies the PDE and the boundary conditions.

b. Now let $w = w(x, t)$ and consider the initial boundary value problem

$$w_t - w_{xx} = f(x, t) \quad 0 \le x \le 2, \quad t > 0$$

$$w(0, t) = w(2, t) = 0 \quad t > 0$$

$$w(x, 0) = w_0 \quad 0 \le x \le 2$$

Write down an *implicit* finite difference scheme for this problem which is *second order in space and time*. Then show that at a fixed time, we are required to solve a linear system $A\vec{w} = \vec{f}$ and explicitly give A and $\vec{f}$.

c. Let $w(x, t)$. Derive a finite difference approximation to $w_{xxx}(x, t)$ using the values $w(x, t)$, $w(x + h, t)$, $w(x - h, t)$ and $w(x + 2h, t)$. Determine the truncation error for your approximation.

4. *Finite Element* (Dr. Burkardt)

Suppose that $u \in H^1(\Omega)$ is a solution of the Poisson equation $-\Delta u = f$ in the domain Ω, and that for some constant $\alpha > 0$, u satisfies the mixed boundary condition $\alpha u + \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

Recall that $H^1(\Omega)$ is the set of functions $v : \Omega \to \mathbb{R}$ such that v and all first derivatives of v are square-integrable over Ω.

(a) Show that u satisfies the weak equation:

$$\int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} u\, v = \int_\Omega f\, v \quad \text{for all } v \in H^1(\Omega)$$

(b) For any $u, v \in H^1(\Omega)$, define:

$$B(u, v) \equiv \int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} u\, v$$

Show that $B(u, v)$ is an inner product on $H^1(\Omega)$.

(c) Use your answer to (b) to show that a solution of the weak equation is unique.