Lectures - Week 10
Introduction to Ordinary Differential Equations (ODES)
First Order Linear ODEs

When studying ODEs we are considering functions of one independent variable, e.g., $f(x)$, where x is the independent variable and f is the dependent variable. An ODE is an equation which tells us something about how the dependent variable changes with respect to the independent variable, i.e., an equation with the derivative of the dependent variable with respect to the independent variable. First lets recall the definition of the derivative of $f(x)$ which we denote $f'(x)$ or equivalently as $\frac{df}{dx}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The term we are taking the limit of is just the slope of the secant line joining the points $(x, f(x))$ and $\big(x + \Delta x, f(x + \Delta x)\big)$. The derivative has a graphical interpretation which is just the slope of the tangent line to the curve $y = f(x)$ at the point x. The derivative has a physical interpretation which is just the instantaneous rate of change of f with respect to x. Often when modeling some physical, chemical, biological, etc. phenomena we know how some quantity changes with time (or spatially) and we want to find the quantity itself. In this case we often are faced with solving an ODE.

In some sense solving an ODE is like performing an integration. For example if we have

$$y'(t) = t$$

then we must find a function whose derivative is t so we know the result by integrating the equation to get $y(t) = \frac{1}{2}t^2 + C$ where C is an arbitrary constant. The solution is not unique because we can add any constant to $\frac{1}{2}t^2$ and its derivative is still t so it satisfied our ODE. To make the solution unique we add an auxiliary condition (which we will see can be an initial or boundary condition); in our problem we could have

$$y'(t) = t \qquad y(0) = 1$$

whose unique solution is then $y(t) = \frac{1}{2}t^2 + 1$. Note that here we needed one auxiliary condition because we integrated once (because we had the first derivative of y). Recall from calculus that unlike differentiation where there are a set of rules, for integration we learned techniques which applied in certain settings. We also found that some integrals could not be evaluated by standard means. This will be similar to the situation of finding analytical solutions to ODEs. There are techniques which can be used for solving certain types of ODEs while analytical solutions for others are not available in a closed form. To identify what techniques work for which ODEs we classify differential equations into various categories.

ODES are classified in several ways.

- **order** - the highest derivative occurring in the equation
- **linearity** - whether the unknown and its derivatives appear linearly or nonlinearly
- **homogeneous/inhomogeneous** - this refers to the existence of a forcing function on the right hand side of the equation
- **Initial value problem (IVP) or Boundary Value Problem (BVP)** - this refers to the type of auxiliary conditions applied
- **single equation or system** - this refers to whether we have a single equation or a system of coupled ODEs.

Example Classify each of the equations using the above criteria.

(1) $$y'(t) + 6y(t) = 7\sin t \quad t > 0, \quad y(0) = 1$$

(2) $$y''(x) + 2y(x) = x^3 \quad 0 < x < 1, \quad y(0) = 1, y(1) = 4$$

(3) $$y'(t) + y^2(t) = 0 \quad t > 2, \quad y(2) = -1$$

(4) $$y'(t) + w(t) = e^t, w'(t) - y(t) = 4 \quad t > 2, \quad y(2) = -1, w(2) = 0$$

In the second equation the notation $y''(x)$ denotes the second derivative of y with respect to x and is also denoted d^2y/dx^2; this just means take the first derivative of the function $y'(x)$.

Equation (1) is a first order, linear inhomogeneous IVP and is of course a single equation. This is an IVP because we are given the value of y initially and want to find it for later times.

Equation (2) is a second order, linear inhomogeneous BVP and is of course a single equation. It is a BVP because we seek $y(x)$ in the interior of the interval $[0, 1]$ and we know the values at the two endpoints of the domain, i.e., its boundaries.

Equation (3) is a first order, nonlinear homogeneous IVP and is of course a single equation. It is nonlinear because the unknown y appears nonlinearly in the term y^2.

Equation(4) is a first order system of linear inhomogeneous equations and is an IVP.

In a course in ODEs one learns a variety of techniques which are applicable to certain types of equations. For example, one technique might be applicable for second order, linear, homogeneous equations. However, one should keep in mind that these techniques only work in specific cases; there are many ODEs which do not have analytical solutions. Our goals for this portion of the course on ODEs are:

- State existence and uniqueness results for IVPs and BVPs; this will require us to look at some concepts such as continuity, differentiability, Lipschitz continuity, convexity.
- Look at a few techniques for finding analytical solutions of ODEs
- Look at the two basic types of numerical methods for approximating the solution of IVPs and be able to analyze accuracy, stability, etc.
- Look at finite difference techniques for solving a BVP
- Compare the implications of numerically solving an IVP and a BVP

First Order Linear ODEs

The simplest ODE to solve is a first order linear ODE which has the general form

(5) $$y'(t) + p(t)y(t) = g(t) \quad t_0 < t < T \qquad y(t_0) = y_0$$

Note that this is an IVP because we are given the value of y initially at $t = t_0$ and want to determine the value of y for later times. Also note that we have assumed the coefficient of $y'(t)$ is one in our general formulation. We do this because we can always divide through by the coefficient if it is not one; for example

$$(\sin t)y'(t) + (\cos t)y(t) = 4 \Rightarrow y'(t) + (\cot t)y(t) = 4/\sin t$$

When we studied numerical methods for solving $A\vec{x} = \vec{b}$ we first wanted to know if the system had a unique solution. The same is true for ODEs – we want to determine conditions which *guarantee* a solution exists and is unique.

Theorem Let $p(t), g(t)$ be continuous functions on (t_0, T). Then the solution to (5) exists and is unique.

Note that the theorem gives conditions under which we are guaranteed a unique solution. It does not say anything about the existence when p or g is not continuous.

Recall that a function $f(x)$ is continuous at a point $x = a$ provided we can make $f(x)$ as close to $f(a)$ as we want by making x close to a. This is stated mathematically in the following definition.

Definition A function $f(x)$ is continuous at $x = a$ if given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever } |x - a| < \delta\,.$$

A function $f(x)$ is continuous in an entire interval $[a, b]$ provided it is continuous at each point in the interval.

Recall that this means graphically that the function has no 'jumps" in its plot, i.e., if we put our pencil down at the point $[a, f(a)]$ we can trace out the entire curve to $[b, f(b)]$ without raising our pencil.

Example Determine if the above existence theorem can be applied to each IVP.

$$y'(t) + \sin \pi t\, y(t) = e^t, y(1) = 1 \quad 1 < t < 4 \qquad (t-2)y'(t) + e^t y(t) = 4, y(1) = 1 \quad 1 < t < 4$$

In the first equation $p(t) = \sin \pi t$ and $g(t) = e^t$ which are both continuous functions in $[1, 4]$ so we are guaranteed that a solution exists and is unique from the theorem. For the second equation $p(t) = e^t/(t - 2)$ and $g(t) = 4$. Although $g(t)$ is continuous on $[1, 4]$, $p(t)$ is not and so the theorem is not applicable.

It turns out that homogeneous equations are not only easier to solve than inhomogeneous ones, but often the solution of the homogeneous equation tells us some part of the solution to the inhomogeneous one. For this reason we first look at techniques for solving homogeneous first order linear ODEs.

Separation of Variables for solving first order linear homogeneous ODEs

We now consider the IVP

$$y'(t) + p(t)y(t) = 0 \quad t_0 < t < T, \qquad y(t_0) = y_0$$

or equivalently

$$y'(t) = -p(t)y(t) \quad t_0 < t < T, \qquad y(t_0) = y_0$$

If $p(t)$ is a constant, this says that the rate of change of y is proportional to itself or equivalently we are asking ourselves to find a function y whose derivative is a constant times itself; we know that the only function which satisfies this is the exponential. For example, if $p(t) = 2$ then we want $y(t)$ such that $y'(t) = -2y(t)$ which implies $y(t) = Ce^{-2t}$. To solve the equation for general $p(t)$ we note that the ODE can be rewritten as

$$\frac{1}{y}y'(t) = -p(t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = -p(t) \Rightarrow \frac{dy}{y} = -p(t)dt$$

In the last equation we see that on the left hand side we only have the dependent variable y and on the right hand side the independent variable t. We can then integrate both sides of the equation to get

$$\int \frac{1}{y}\, dy = -\int p(t)\, dt \Rightarrow \ln|y| + C_1 = -\int (p(t)\, dt$$

Solving for y yields

$$y(t) = e^{-\int (p(t)\, dt - C_1} = e^{-\int (p(t)\, dt} e^{-C_1} = Ce^{-\int (p(t)\, dt}$$

This is not a formula that we should memorize but rather remember how it was obtained by separation of variables and do this for each problem. Also one should remember that the solution of our first order linear homogeneous ODE was in the form of an exponential.

Example Use separation of variables to solve the IVP

$$y' - 2ty = 0 \quad 0 < t < 4 \qquad y(0) = 4$$

Separating variables yields

$$\frac{y'}{y} = 2t \Rightarrow \int \frac{dy}{y} = \int 2t\, dt \Rightarrow \ln|y| + C_1 = t^2 + C_2 \Rightarrow y(t) = Ce^{t^2}$$

Satisfying the initial condition $y(0) = 4$ gives $y(0) = Ce^0 = C$ implies $C = 4$ so the solution is $y(t) = 4e^{t^2}$. Note that we really don't have to use two arbitrary constants here because they sum to another arbitrary constant. It would be OK to say $\ln|y| = t^2 + \tilde{C}$ implies $y(t) = Ce^{t^2}$. Also note that once we have our solution, we can verify that it is correct by differentiating and demonstrating that it satisfies the given ODE

$$y(t) = 4e^{t^2}, \quad y'(t) = 8te^{t^2} \Rightarrow y' - 2t = 8te^{t^2} - 2(4e^{t^2}) = 0$$

Also $y(0) = 4e^0 = 4$ so the initial condition is satisfied.

Example Use separation of variables to solve the IVP

$$y' - \sin t^2 y = 0 \quad 0 < t < 4$$

Separating variables yields

$$\frac{y'}{y} = \sin t^2 \Rightarrow \int \frac{dy}{y} = \int \sin t^2 \, dt \Rightarrow \ln|y| + C_1 = \int \sin t^2 \, dt \Rightarrow y(t) = Ce^{\int \sin t^2 \, dt}$$

Note that although we have a formula for the solution it is in terms of an integral because we don't have an anti-derivative of $\sin t^2$. To use this formula we would have to approximate the integral.

All first order linear homogeneous ODEs can be solved via this technique but we can't always get a closed form solution as in the previous example because we can't always integrate $\int p(t)\, dt$.

Integrating factors for solving first order linear inhomogeneous ODEs

If the right hand side of our ODE is not zero, then separation of variables does not work. For example,

$$y'(t) - ty(t) = 4 \Rightarrow \frac{dy}{y} = (t + 4/y)dt$$

which is not separated. However, there is a technique called an *integrating factor* that makes it possible to solve the general ODE given in (5) but as before the solution may involve integrals which we are unable to evaluate. The idea is that we want to find a function which we can multiply the equation by so we can integrate it. The following example illustrates the technique.

Example Consider the ODE

$$y'(t) + (\sin t)y(t) = 4$$

Multiply the equation by $e^{-\cos t}$ and integrate to find $y(t)$.

We have

$$e^{-\cos t}\Big[y'(t) + \sin t y(t)\Big] = 4e^{-\cos t}$$

We note that the left hand side of this equation can be written as

$$\frac{d}{dt}\Big[e^{-\cos t}y\Big]$$

because using the product rule gives

$$\frac{d}{dt}\Big[ye^{-\cos t}\Big] = y\big[e^{-\cos t}\big(\sin t\big)\big] + y'(t)e^{-\cos t}\,.$$

Thus we have

$$\frac{d}{dt}\Big[e^{-\cos t}y\Big] = 4e^{-\cos t} \Rightarrow \int d\Big[e^{-\cos t}y\Big] = \int 4e^{-\cos t}dt$$

which yields

$$e^{-\cos t}y = 4\int e^{-\cos t}dt \Rightarrow y(t) = 4e^{\cos t}\int e^{-\cos t}dt$$

It turns out that there is a general form of the integrating factor which will work for the ODE given in (5). We simply multiply the equation by

$$e^{\int p(t)\,dt}$$

Why does this work? Because we have

$$y'e^{\int p(t)\,dt} + p(t)ye^{\int p(t)\,dt} = g(t)e^{\int p(t)\,dt}$$

and

$$\frac{d}{dt}\Big[ye^{\int p(t)\,dt}\Big] = y'(t)e^{\int p(t)\,dt} + y(t)\Big[p(t)e^{\int p(t)\,dt}\Big] = e^{\int p(t)\,dt}\Big[y'(t) + p(t)y(t)\big]$$

so the equation becomes

$$\frac{d}{dt}\Big[ye^{\int p(t)\,dt}\Big] = g(t)e^{\int p(t)\,dt} \Rightarrow \int d\Big[ye^{\int p(t)\,dt}\Big] = \int \Big[g(t)e^{\int p(t)\,dt}\Big]dt$$

which implies

$$ye^{\int p(t)\,dt} = \int \Big[g(t)e^{\int p(t)\,dt}\Big]dt \Rightarrow y(t) = e^{-\int p(t)\,dt}\int \Big[g(t)e^{\int p(t)\,dt}\Big]dt\,.$$

Example Use an integrating factor to solve

$$y'(t) + 2ty(t) = 4t \quad y(0) = 5$$

For this problem the integrating factor is $e^{\int 2t} = e^{t^2}$; note that we didn't have to include the arbitrary constant of integration. Why? We have

$$e^{t^2} y'(t) + 2tye^{t^2} = 4te^{t^2}$$

As a check we note that

$$\frac{d}{dt}\left[e^{t^2} y\right] = e^{t^2} y'(t) + 2tye^{t^2}$$

so our equation becomes

$$\int d\left[e^{t^2} y\right] = \int 4te^{t^2}\ dt$$

Now the integral on the right hand side can be evaluated by substitution, i.e., letting $u = t^2, du/dt = 2t$ then we have

$$4\int te^{t^2}\ dt = 4\int e^u \frac{1}{2} du = 2e^u + C = 2e^{t^2} + C$$

Thus we have

$$e^{t^2} y = 2e^{t^2} + C \Rightarrow y(t) = 2 + Ce^{-t^2}$$

Satisfying the initial condition $y(0) = 5$ gives $y(t) = 2 + Ce^0 = 2 + C = 5$ implies $C = 3$. As a check you should verify that this solution satisfies the IVP.

Note that integrating factors work when solving homogeneous equations too. For example if we return to the problem

$$y' - 2ty = 0$$

we multiply through by $e^{-\int 2tdt} = e^{-t^2}$ to get

$$e^{-t^2} y' - 2te^{-t^2} y = 0 \Rightarrow \int d(e^{-t^2} y) = \int 0\ dt \Rightarrow e^{-t^2} y = C \Rightarrow y = Ce^{t^2}$$