Multigrid solvers

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May 12 – June 19, 2014

Multigrid

Multigrid

Geometrical multigrid

- Simple iterative methods tend to damp high (spatial) frequency errors fast.
- After a few smoothing steps of a simple method, map the current error out to a coarser grid.
- Errors will have relatively higher spatial frequency there.
- Take a few more steps of a simple method on the coarser grid.
- Continue mapping to coarser grids until grid is coarse enough to solve.
- Interpolate back to the next finer grid and do few smoothing steps
- Continue to the finest grid
- Repeat until converged.

Advantages of GMG

- Number of iterations should not depend on number of mesh points!
- Works very well as preconditioner for Krylov methods

Gauss-Seidel iterations

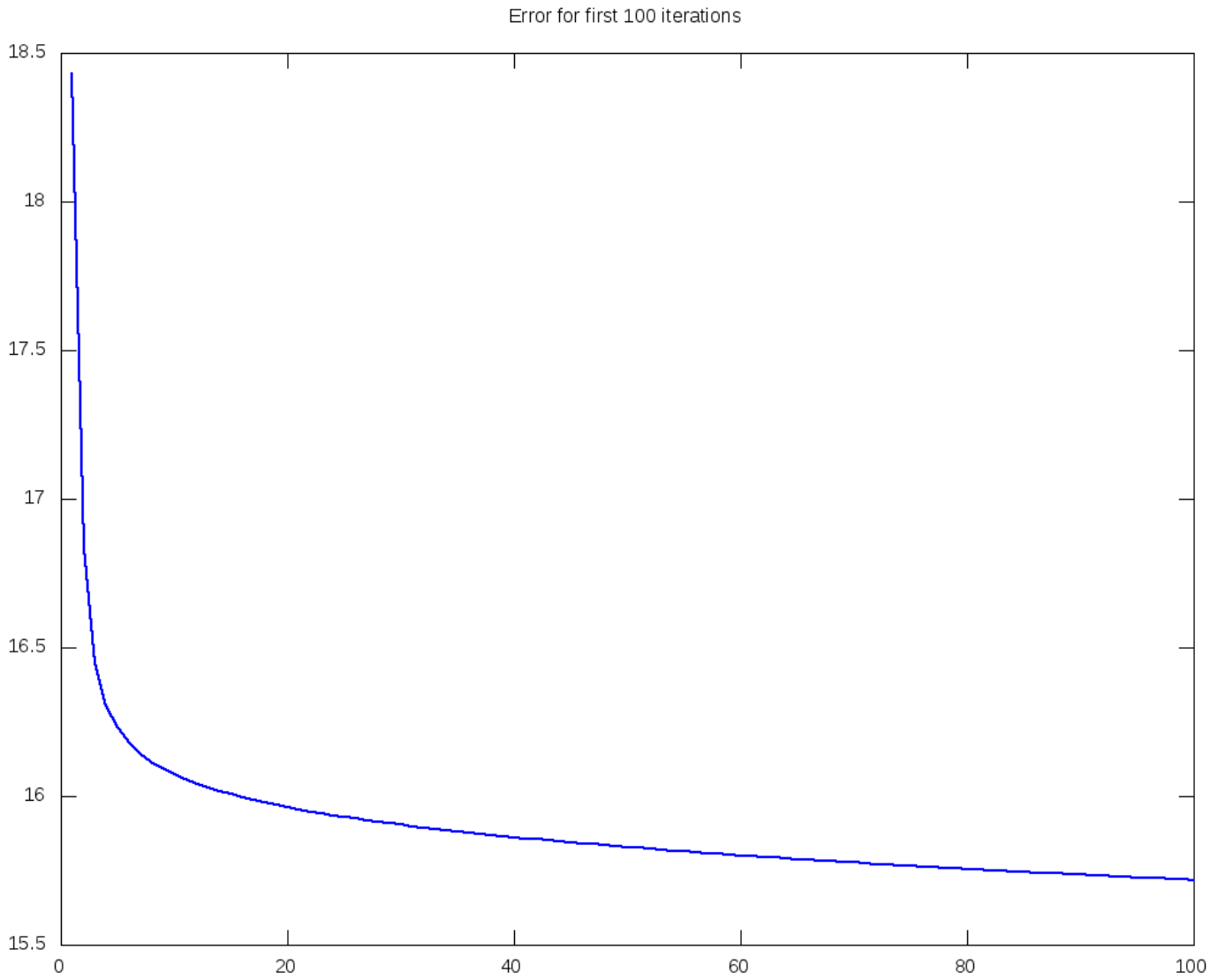
To solve an $n \times n$ matrix system,

$$Au = f$$

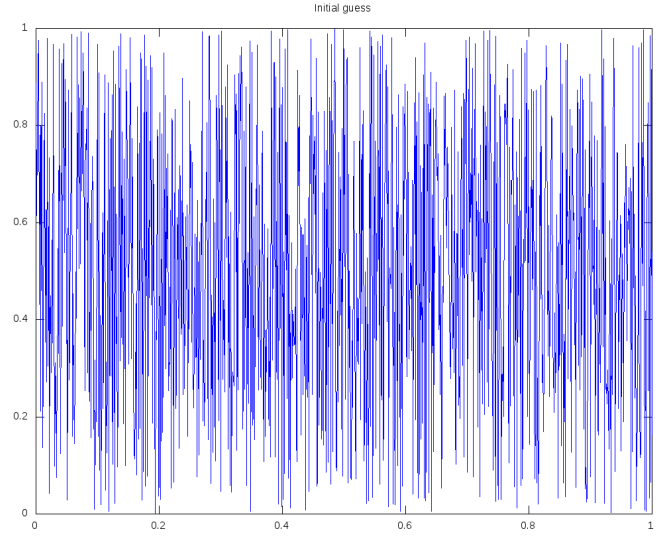
given an initial guess $u^{(0)}$, for $k = 1, 2, \ldots,$ set

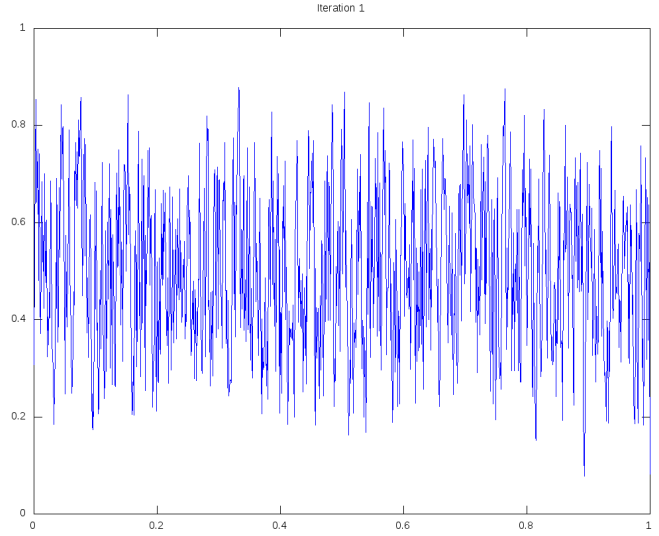
$$u_i^{(k+1)} = \left(f_i - \sum_{j=1}^{i-1} A_{ij} u_j^{(k+1)} - \sum_{j=i+1}^{n} A_{ij} u_j^{(k)} \right) / A_{ii}$$

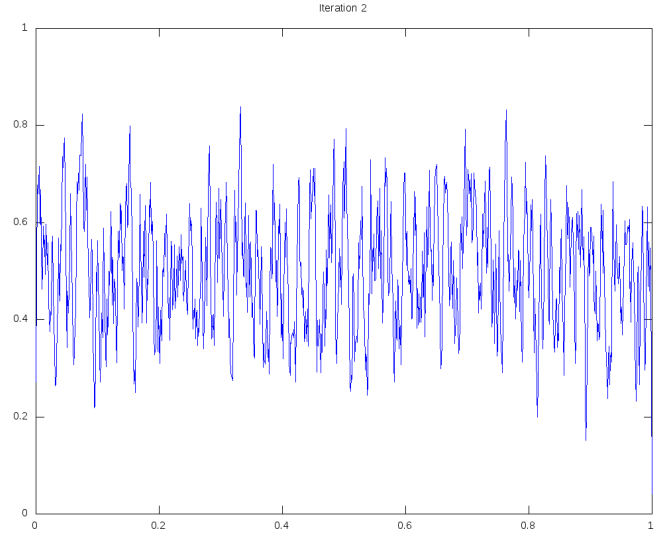
Gauss-Seidel starts fast, slows down

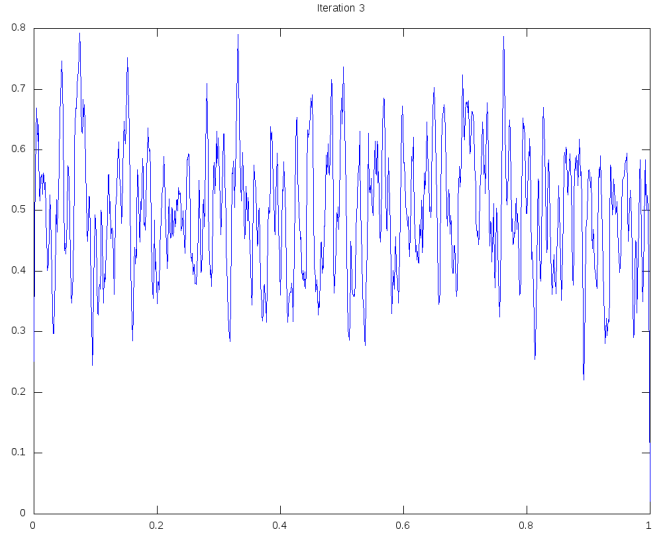


Error gets smooth fast









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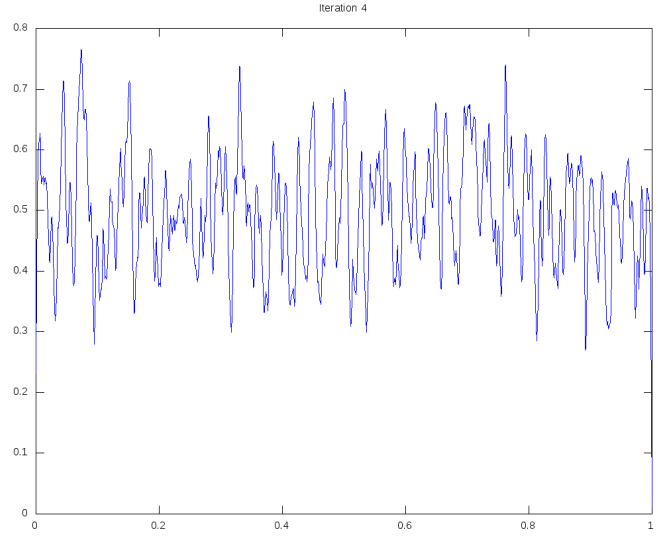
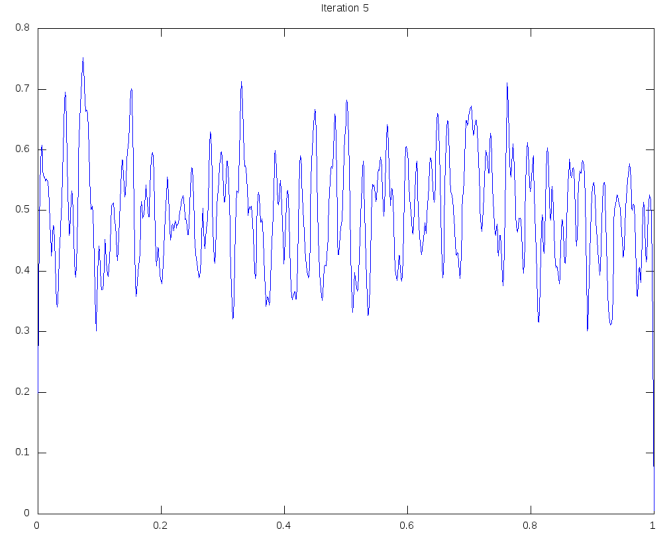
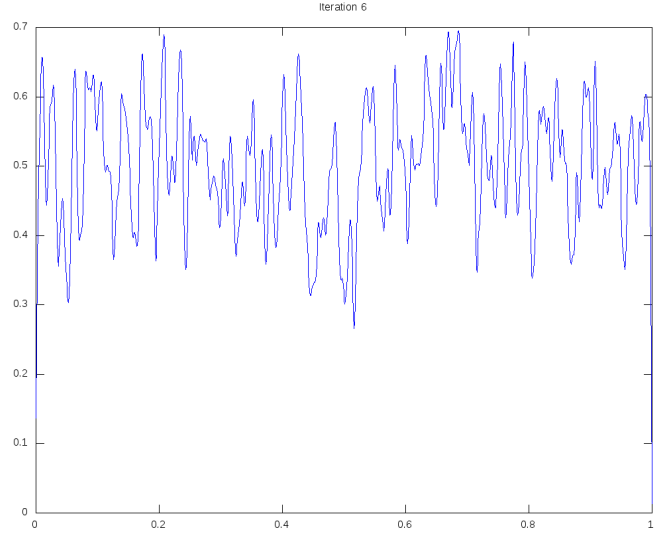
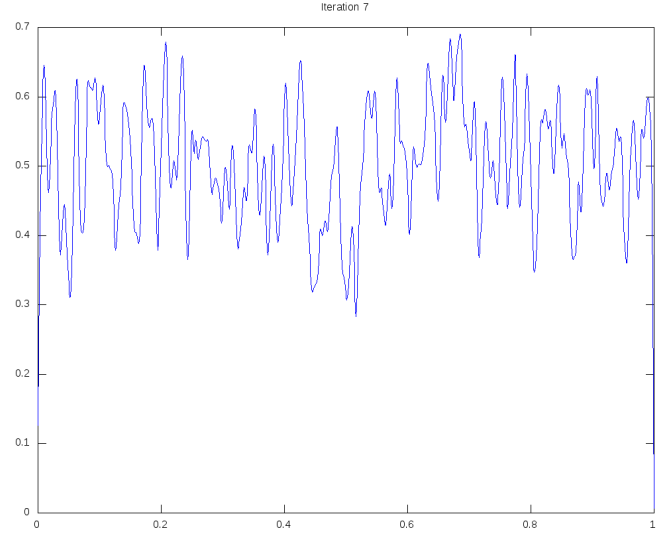
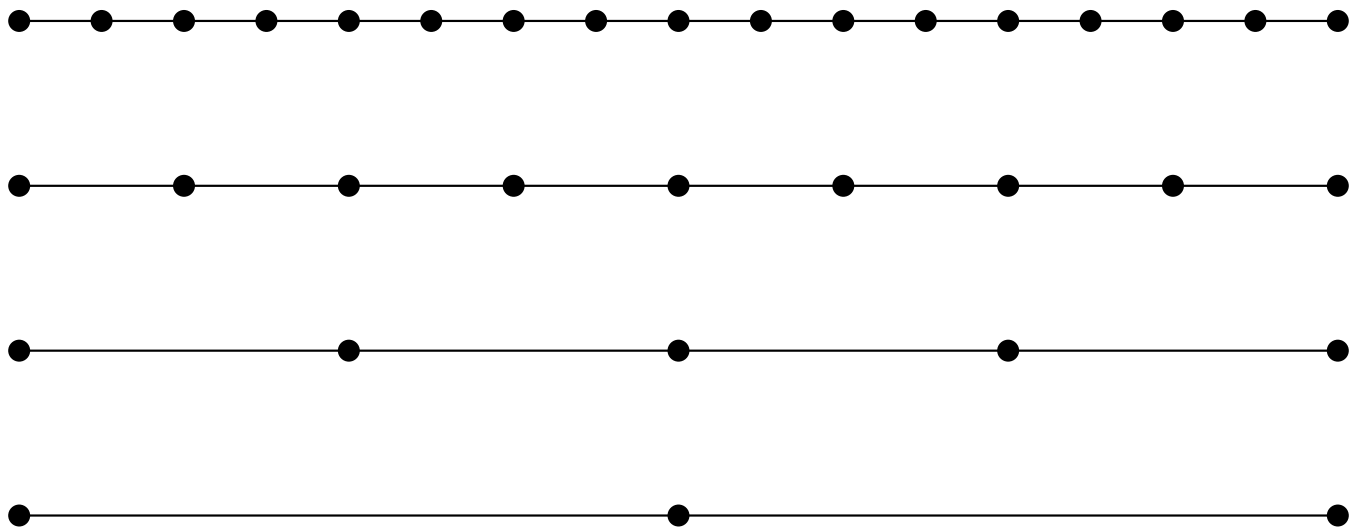








Multiple 1D grids



Interpolation or prolongation

If a solution is known on a grid, how should it be transferred to the next finer grid?

- For fine grid points that agree with coarse points, copy.
- For fine grid points between two coarse points, average.

Interpolation matrix 5 pts to 9 pts

$$P_{9\times 5} = \begin{pmatrix} 1 & & & & \\ .5 & .5 & & & \\ & 1 & & & \\ & .5 & .5 & & \\ & & 1 & & \\ & & .5 & .5 & \\ & & & 1 & \\ & & & .5 & .5 \\ & & & & 1 \end{pmatrix}$$

Restriction

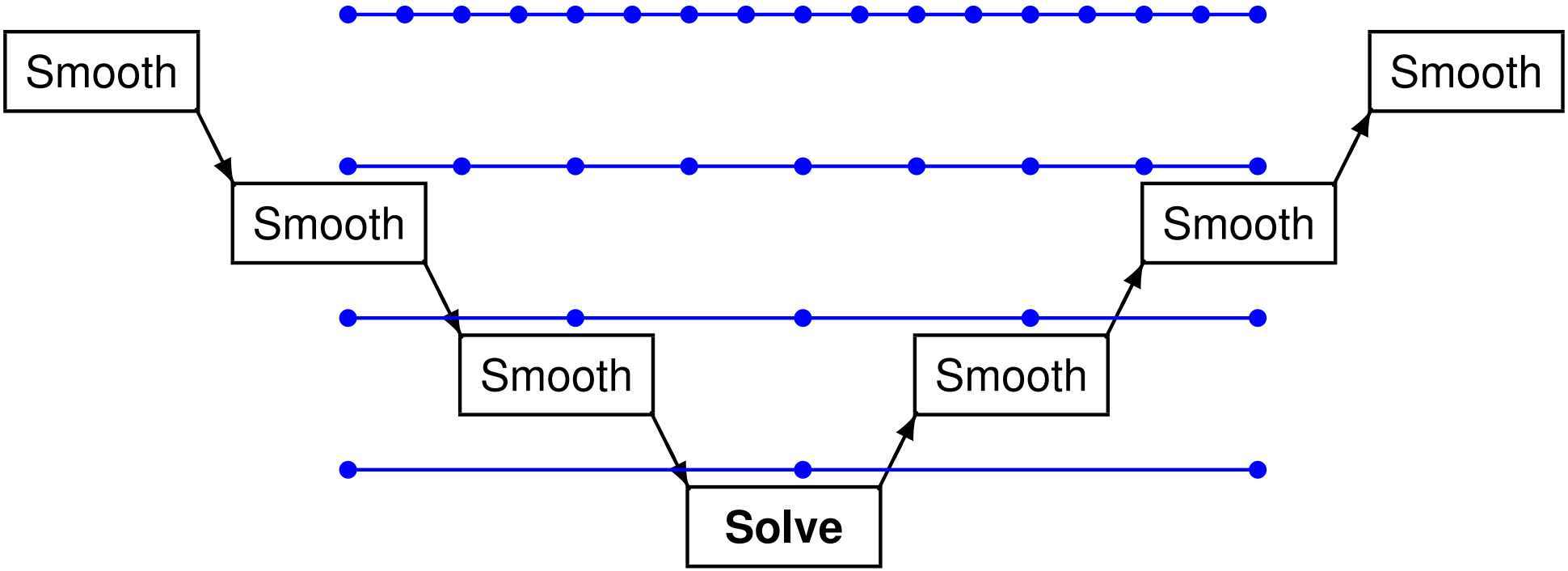
If a solution is known on a fine grid, how should it be transferred to the next *coarser* grid?

$$P_{5\times 9} = (P_{9\times 5})^T$$

- Maintain symmetry!
- Proofs fail without it!
- It works better this way.

Multigrid

The V-cycle



The V-cycle: Python code

```
def vcycle(A,f):
    # perform one v-cycle on the matrix A

    sizeF = np.size(A,axis=0);
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    # size for direct inversion < 15
    if sizeF < 15:
        v = la.solve(A,f)
        return v
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    # N1=number of Gauss-Seidel iterations before coarsening
    N1 = 5;
    v = np.zeros(sizeF);
    for numGS in range(N1):
        for k in range(sizeF):
            v[k] = (f[k] - np.dot(A[k,0:k], v[0:k]) \
                     -np.dot(A[k,k+1:], v[k+1:]) ) / A[k,k];
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                    -np.dot(A[k,k+1:], v[k+1:]) ) / A[k,k];

    # construct interpolation operator from next coarser to this mesh
    # next coarser has ((n-1)/2 + 1 ) points
    assert(sizeF%2 ==1)
    sizeC =  (sizeF-1)/2 +1
    P = np.zeros((sizeF,sizeC));
    for k in range(sizeC):
        P[2*k,k] = 1;   # copy these points
    for k in range(sizeC-1):
        P[2*k+1,k] = .5;    # average these points
        P[2*k+1,k+1] = .5;
```

The V-cycle: Python code cont'd

```
# compute residual
residual = f - np.dot(A,v)
```

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```
# compute residual
residual = f - np.dot(A,v)

# project residual onto coarser mesh
residC = np.dot(P.transpose(),residual)
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```
# compute residual
residual = f - np.dot(A,v)

# project residual onto coarser mesh
residC = np.dot(P.transpose(),residual)

# Find coarser matrix  (sizeC X sizeC)
AC = np.dot(P.transpose(),np.dot(A,P))
```

The V-cycle: Python code cont'd

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# compute residual
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vC = vcycle(AC,residC);
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residC = np.dot(P.transpose(),residual)

# Find coarser matrix  (sizeC X sizeC)
AC = np.dot(P.transpose(),np.dot(A,P))

vC = vcycle(AC,residC);

# extend to this mesh
v = np.dot(P,vC)
```

The V-cycle: Python code cont'd

```
# compute residual
residual = f - np.dot(A,v)

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residC = np.dot(P.transpose(),residual)

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AC = np.dot(P.transpose(),np.dot(A,P))

vC = vcycle(AC,residC);

# extend to this mesh
v = np.dot(P,vC)

# N2=number of Gauss-Seidel iterations after coarsening
N2 = 5;
for numGS in range(N2):
    for k in range(sizeF):
        v[k] = (f[k] - np.dot(A[k,0:k], v[0:k]) \
                  -np.dot(A[k,k+1:], v[k+1:]) ) / A[k,k];
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The V-cycle: Python code cont'd

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# compute residual
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return v
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Solving with V-cycles `gmgsolve.py`

```
N = 2**9+1
x = np.linspace(0,1,N);
h = x[1]-x[0]
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Solving with V-cycles `gmgsolve.py`

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N = 2**9+1
x = np.linspace(0,1,N);
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# tridiagonal matrix
A = np.diag(2.*np.ones(N)) - np.diag(np.ones(N-1),  1)
A = A/h**2

f = np.ones(N, dtype=float)  #rhs
```

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A = A/h**2

f = np.ones(N, dtype=float)  #rhs

udirect = la.solve(A, f)  # correct solution

u = np.zeros(N) # initial guess
for iters in range(100):
    r = f - np.dot(A,u)
    if la.norm(r)/la.norm(f) < 1.e-10:
      break
    du = vcycle(A, r)
    u += du

    print "step %d, rel error=%e"% \
        (iters+1, la.norm(u-udirect)/la.norm(udirect) )
```

Iterations and problem size

Number of iterations is independent of problem size!

Grid size	Number of iterations
33	20
65	21
129	22
257	22
513	22
1025	22
2049	22

Multigrid

What is needed for MG?

1. Sequence of grids
2. Intergrid transfer operators
3. Smoothing operator
4. Solver for coarsest grid

Multigrid

Multigrid

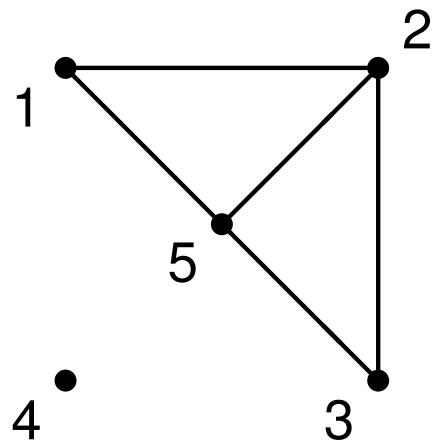
References

- Ruge, J. W., Stüben, K., “Algebraic Multigrid,” Chapt. 4 in McCormick, S. G., *Multigrid Methods*, *Frontiers in Applied Mathematics*, Vol. 4, SIAM, 1987.
- Briggs, W. L., Hensen, V. E., McCormick, S. F., *A Multigrid Tutorial*, Second Edition, SIAM, 2000.
- Trottenberg, U., Oosterlee, C. W., Schüller, A., *Multigrid*, Appendix A by Stüben, K., Academic Press, 2001.

What is a “grid”?

- Every matrix has an assoicated graph

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$



- Given a matrix, the finest grid is its associated graph.

Multigrid

What does “smooth” mean?

- In GMG, we have a notion of “smooth” error and notice that Gauss-Seidel iteration makes rough errors smoother.
- Gauss-Seidel makes very rough errors smooth rapidly, then stalls.

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- In GMG, we have a notion of “smooth” error and notice that Gauss-Seidel iteration makes rough errors smoother.
- Gauss-Seidel makes very rough errors smooth rapidly, then stalls.
- In AMG, we *define* a “rough” error as one that Gauss-Seidel is effective in reducing and a “smooth” error as one on which Gauss-Seidel stalls.
- Loosely speaking, an error is “smooth” when $Ae \approx 0$.
- $a_{ii} e_i \approx -\sum_{i \neq j} a_{ij} e_j$

Simplifying assumption

From now on, **assume that the matrix A is a symmetric M-matrix.**

1. Diagonal elements are positive, off-diagonal are 0 or negative
2. Diagonal $>$ $-$(sum of off-diagonals)

- Original work on AMG was done for M-matrices.
- Some proofs are possible.

How to construct a coarse grid from a fine one.

- Define the notion of “strong dependence” (“influence”, “coupling”).
- Break the mesh up into regions in which each point is strongly dependent on a few distinguished points.
- The distinguished points will be the coarse mesh points.
- The coarse-to-fine mesh interpolation will be based on strong dependence.

Strong dependence

Def. 1 Given a threshold $0 < \theta \leq 1$, the variable u_i "strongly depends" on the variable u_j if

$$-a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\}$$

Def. 2 If the variable u_i strongly depends on the variable u_j, then the variable u_j "strongly influences" u_i.

Important feature of strong dependence

- Smooth error varies slowly in the direction of strong connection
- (See the discussion in Briggs, Henson, McCormick)

Coarsening

- Suppose you have a given fine grid
- Divide into C-points and F-points
- C-points will be next coarser grid

Coarsening

Requirements for C-points include

- Smooth error can be approximated accurately
- Smooth functions can be interpolated accurately
- Substantilly fewer points

Definitions

- Neighborhood N_i is the set of all points j with $a_{ij} \neq 0$
- S_i is the set of all points that strongly influence i
- C_i is the set of C-points that strongly influence i

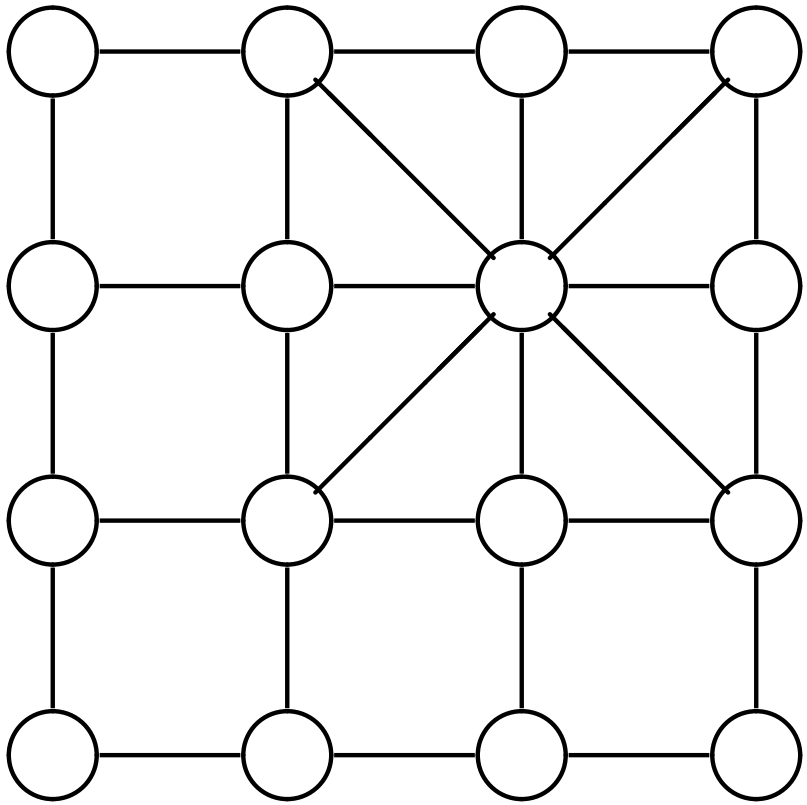
Coarsening heuristics

H-1 For each F-point i, every point $j \in S_i$ that strongly influences i either should be in C_i or should strongly depend on at least one point in C_i

H-1a (Aggressive coarsening) For each F-point i, every point $j \in S_i$ that strongly influences i either should be in C or should strongly depend on at least one point in C

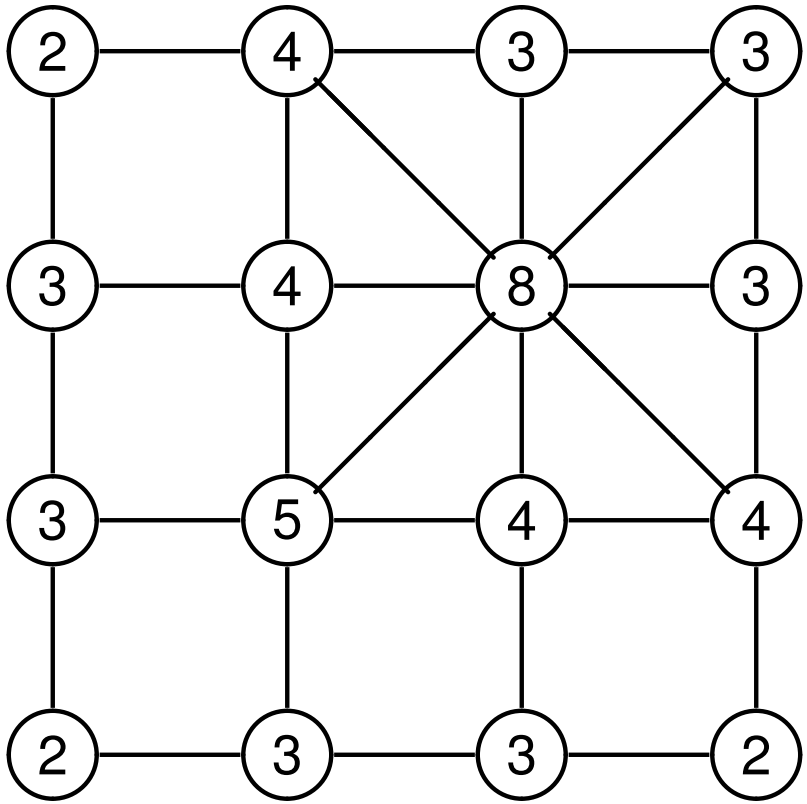
H-2 The set of all coarse points C should be a maximal subset of all points with the property that no C-point strongly depends on another C-point.

Example



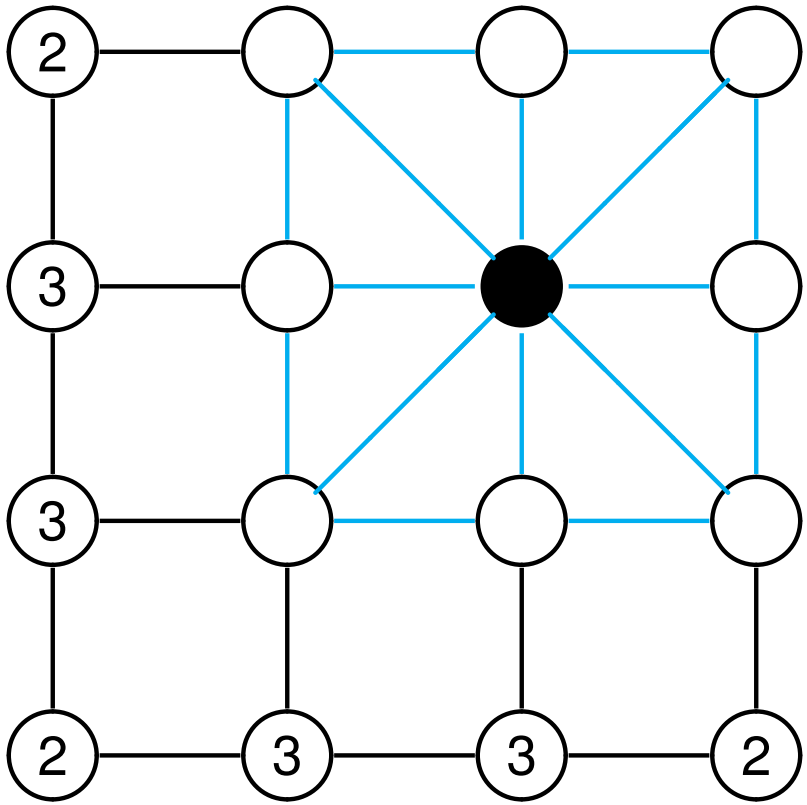
Mesh with strong couplings

Example



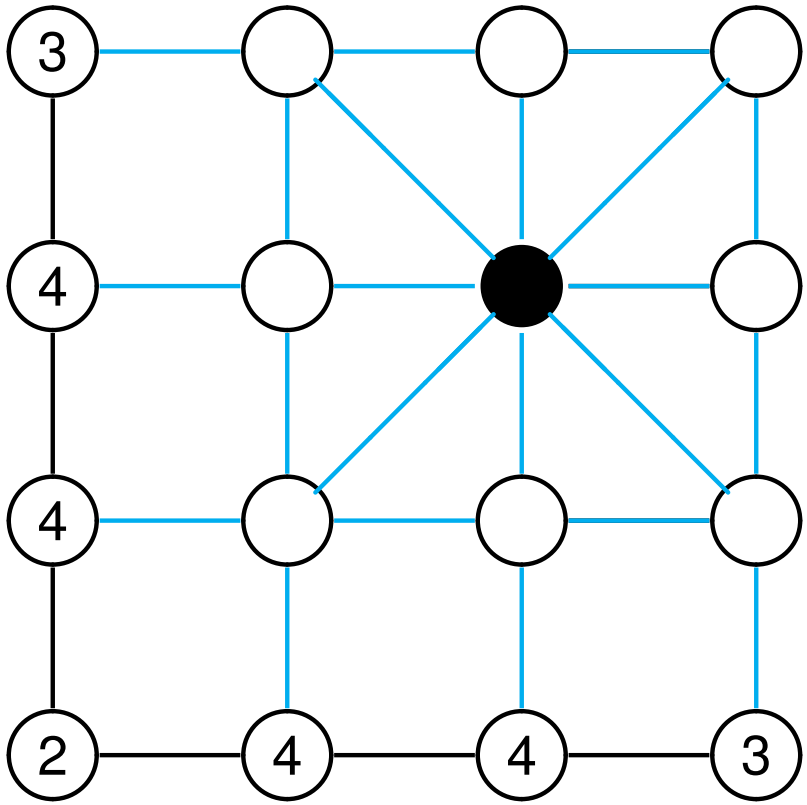
Values

Example



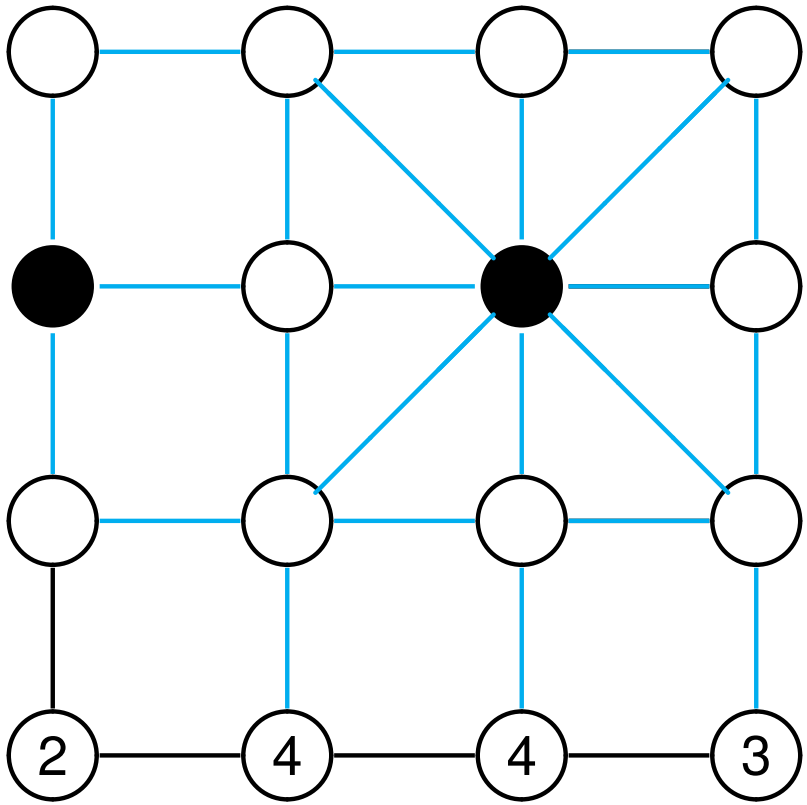
C and F points

Example



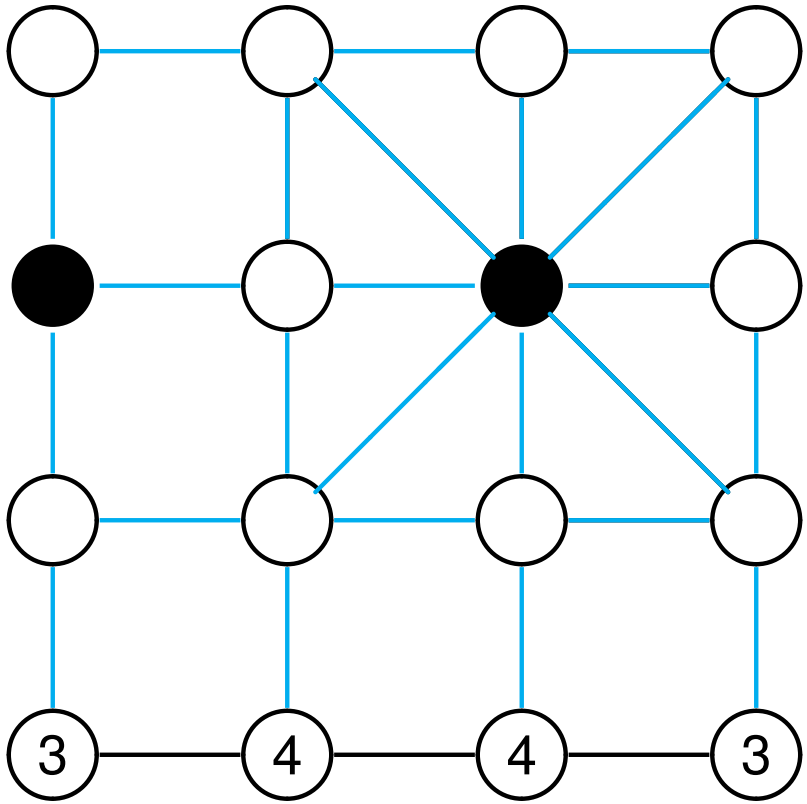
Increment remaining values

Example



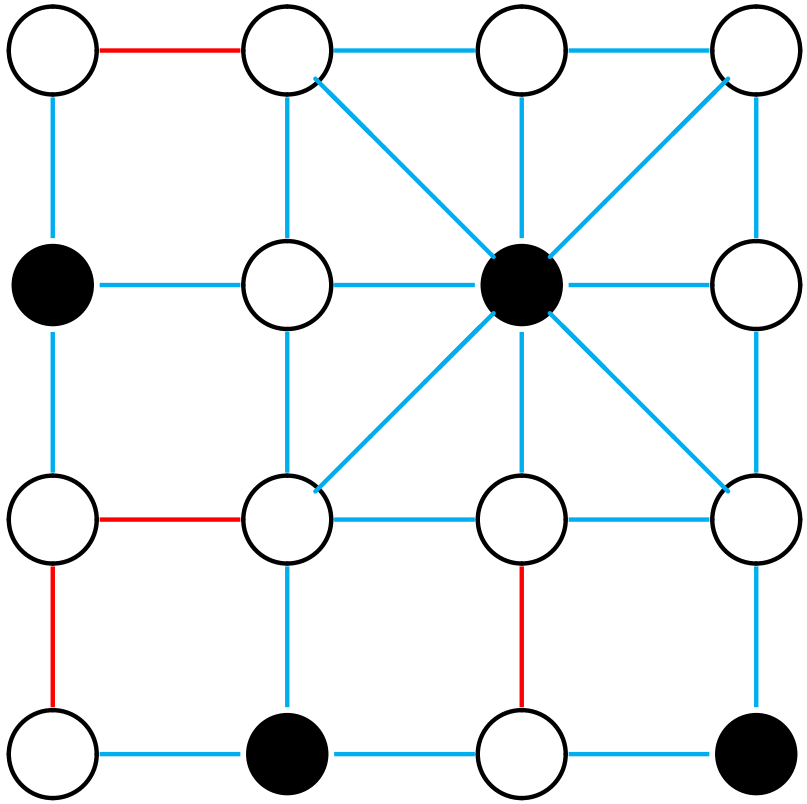
Pick another C point

Example



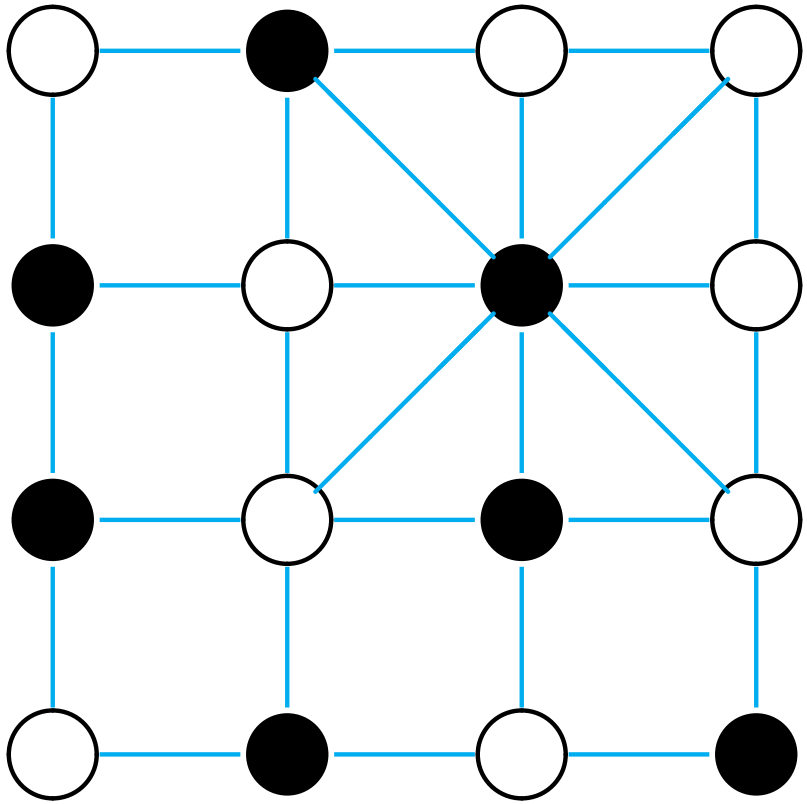
Increment value

Example



Two more C points
Hypothesis 1 failures in red
Aggressive coarsining finishes here.

Example: final (standard) coarsening



Aggressive Coarsening

- Results in a coarser mesh but slower convergence.
- Can be using on only some levels
- Requires a different interpolation formula, with longer-range couplings

Interpolation from F to C

- Want

$$(P_{C\times F}e)_i = \begin{cases} e_i & i \in C \\ \sum_{j\in C_i} w_{ij} e_j & i \in F \end{cases}$$

- Error is smooth on F $\implies$ residual is small

$$a_{ii} \approx -\sum_{j\in N_i} a_{ij} e_j$$

- N_i^S is strongly-coupled F points, N_i^W is weakly

$$a_{ii} \approx -\sum_{j\in S_i} a_{ij} e_j - \sum_{j\in N_i^S} a_{ij} e_j - \sum_{j\in N_i^W} a_{ij} e_j$$

- Put weakly-coupled F points into diagonal

$$(a_{ii} + \sum_{j\in N_i^W} a_{ij}) e_i \approx -\sum_{j\in S_i} a_{ij} e_j - \sum_{j\in N_i^S} a_{ij} e_j$$

Strongly-coupled F points get distributed

- Distribute N_i^S points to all of S_i. For $j \in N_i^S$,

$$e_j \approx \frac{\sum_{k \in C_i} a_{jk} e_k}{\sum_{k \in C_i} a_{jk}}$$

- Hence

$$w_{ij} = -\frac{a_{ij} + \sum_{m \in N_i^S} \left(\frac{a_{im} a_{mj}}{\sum_{k \in C_i} a_{mk}} \right)}{a_{ii} + \sum_{n \in N_i^W} a_{in}}$$