Integration of Logarithmically Smooth Functions over Large Domains

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We seek to explore the properties of logarithmically smooth functions, in particular their behavior with integration. The focus is on the exploration of logarithms to assist in overcoming slow convergence (as the number of points n approaches ∞) of integrals calculated with relatively simple methods such as the Trapezoid Rule. Usually, the kinds of functions that misbehave in simpler methods are those that contain boundaries that span multiple orders of magnitude, i.e., an integral of the form

$$I = \int_D f(x)\, dx$$

such that D is a range for $x \in \mathbb{R}$ such that $10^{-3} \leq x \leq 10^9$. For this example, the integral defined as

$$I = \int_D -\frac{1}{x^2}\, dx$$

where D is defined as above, will be analyzed. Such a problem can cause precision error due to the analytical solution:

$$I = \left[\frac{1}{x}\right]_{10^{-3}}^{10^9}$$

$$\Rightarrow \boxed{I = -999.999999999}$$

whereas the approximation from the Trapezoid Method returns the results as shown in the table below.

Num. of points 10^n	Approximation	Absolute Error
$n = 1$	-5.5556e+13	-5.5556e+12
$n = 2$	-5.0505e+12	-5.0505e+11
$n = 3$	-5.0050e+11	-5.0050e+10
$n = 4$	-5.0005e+10	-5.0004e+09
$n = 5$	-5.0001e+09	-5.0000e+08
$n = 6$	-5.0000e+08	-4.9999e+07
$n = 7$	-5.0000e+07	-4.9999e+06
$n = 8$	-5.0000e+06	-4.9990e+05

Using Log-Space Points

To mitigate this trend, one approach is to use logarithmically-spaced points. In a language like MATLAB, this is defined in the method `logspace`. In this example, the integral being analyzed is defined by a problem optimized for the Gauss-Hermite Quadrature

$$I = \int_0^\infty x \cdot e^{-x^2}$$

whose analytical solution is equal to 0.5. After adapting the boundaries for log-space, Figure 1 shows a higher convergence rate, and the computational cost increase was non-existent.

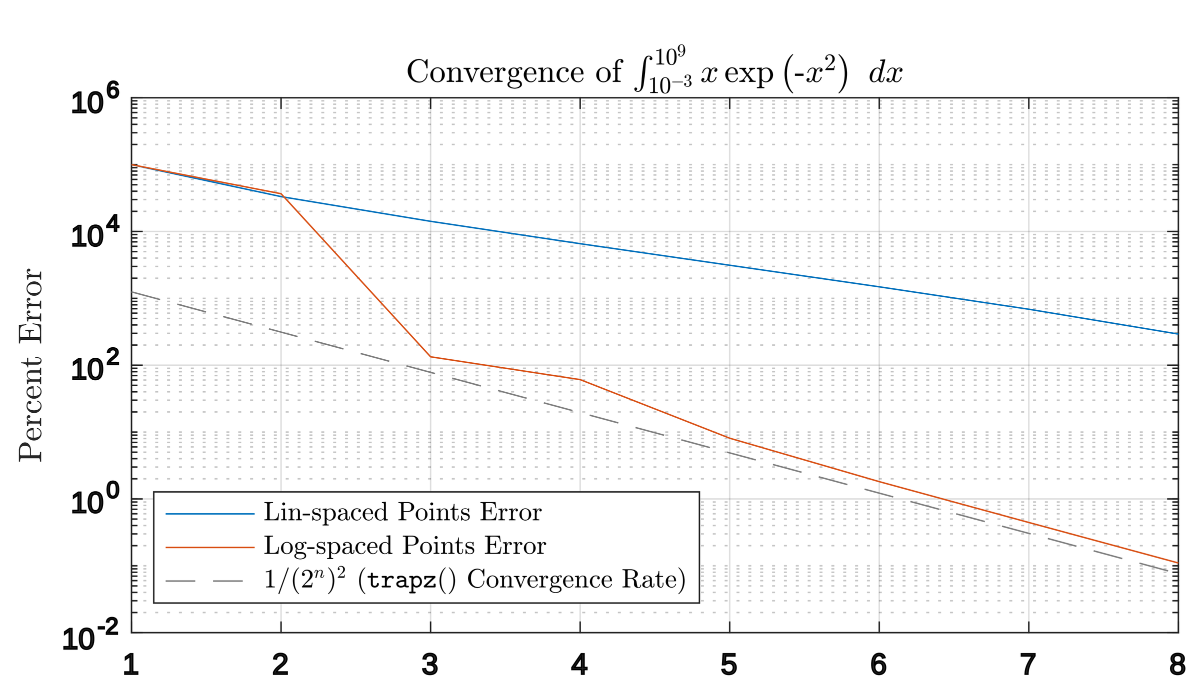

Figure 1: Convergence of `trapz` using linearly-spaced points vs. logarithmically-spaced points as $n \to \infty$.

Adapting Analytically (Log-Log)

Another adaptation that is possible is to adapt the problem analytically to logarithmic space. It would involve adapting the equation as $\log y = m \log x + n \Rightarrow y = e^n x^m$ [1]. This new method, called `LogTrapz1`, adapts the Trapezoid Method into a new summation

$$I = \sum_{i=0}^{n_p - 1} f(x_i, x_{i+1})$$

where $f(x_i, x_{i+1})$ is defined as

$$f(x_i, x_{i+1}) = \begin{cases} e^n \cdot \left(x_{i+1}^{m+1} - x_i^{m+1}\right) / (m+1) & \text{for } m \neq -1 \\ e^n \log\left(x_{i+1} / x_i \right) & \text{for } m = 1 \end{cases},$$

and m and n are [2] [3]

$$m = \frac{\log y_{i+1} - \log y_i}{\log x_{i+1} - \log x_i} \qquad n = \log y_i - m \log x_i$$

such that $\mathbf{x}, \mathbf{y}$ are of length n_p defined as $\{x_0, x_1, \cdots, x_{n_p - 1}, x_{n_p}\}$, and $\{y(x_0), y(x_1), \cdots, y(x_{n_p-1}), y(x_{n_p})\}$, respectively. The error analysis plot in Figure 2 shows that after 10^8 points, the linearly-spaced points undoubtedly begin to perform better than the logarithmically-spaced points (Figure 2).

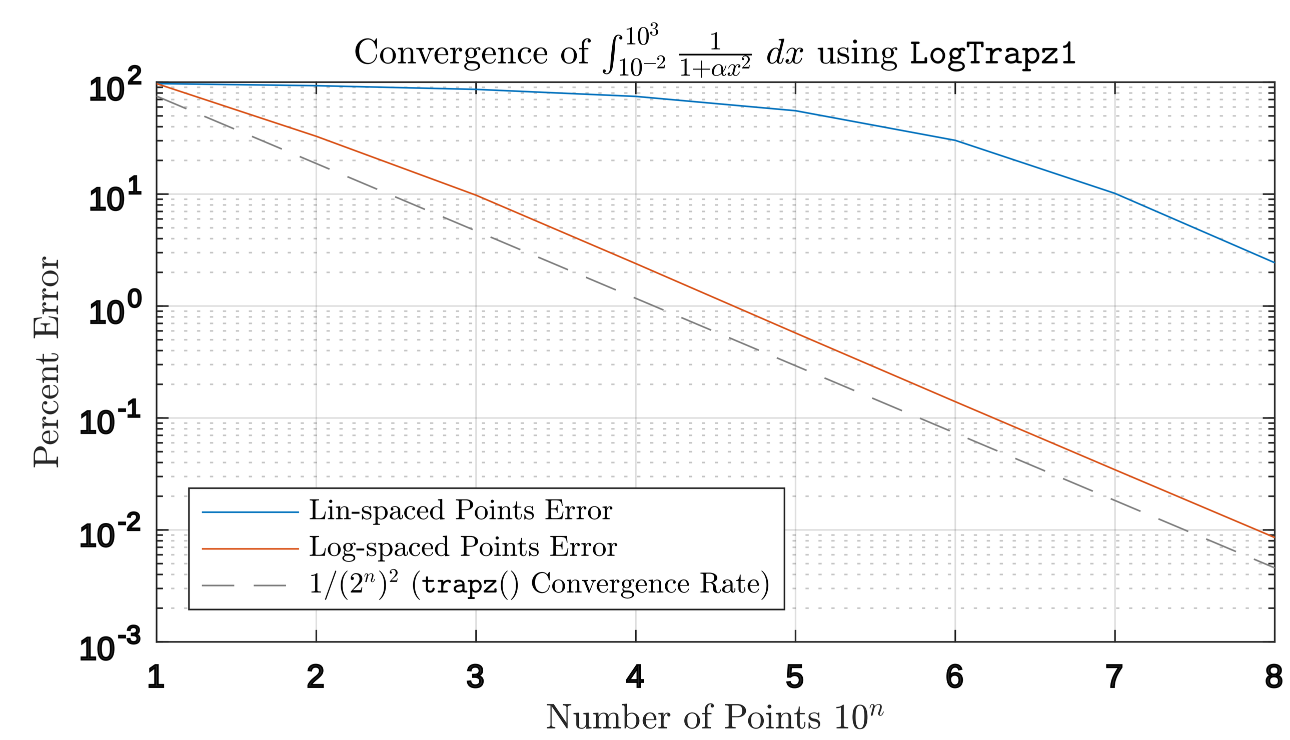

Figure 2: Convergence of `LogTrapz1` using linear-space vs. logarithmic-space as $n \to \infty$.

Adapting Analytically (Log-Lin)

Where log-log space adaptation may fail, adapting to log-lin space may perform better. This method, called `LogTrapz2`, involves adaptation from the same process except adapting into $y = ae^b x \Rightarrow \log y = \log a + bx$ such that

$$a = \log y(x_i) \cdot e^{-bx_i} \qquad b = \frac{\log y_{i+1} - \log y_i}{x_{i+1} - x_i}$$

instead. The resulting plot, shown in Figure 3, shows similar performace to `LogTrapz1` except that the computational cost was less (∼100 seconds on the former vs. ∼75 on the latter using a 2019 MacBook Pro).

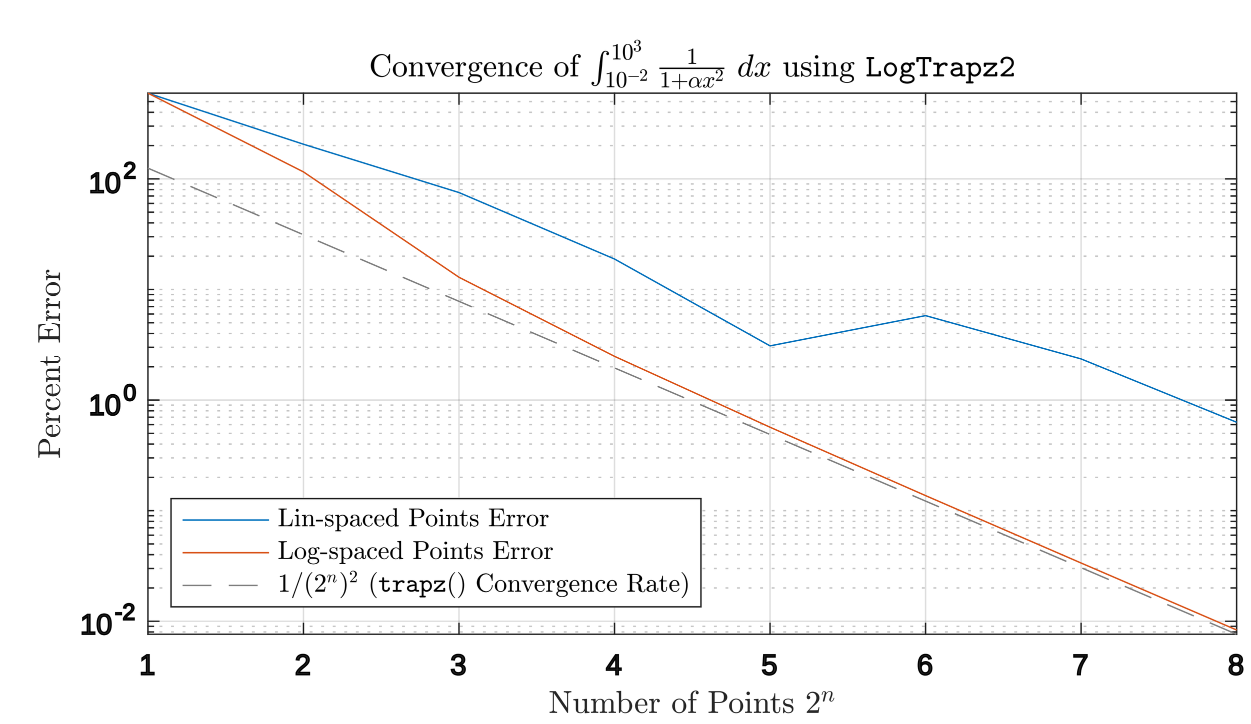

Figure 3: Convergence of `LogTrapz2` using linear-space vs. logarithmic-space as $n \to \infty$.

[1] Shanbhag, S. Evaluating Integrals of Functions with Smoothness Properties in Log-Log Space Using Trapezoidal Rule. `https://gist.github.com/shane5ul/06d000b066636dbaf111187bb4264a2f`.

[2] a user. Feb. 2000. The Origin Forum - Integrating log-log data?. `https://my.originlab.com/forum/topic.asp?TOPIC_ID=1251`.

[3] Pascal, E. 2015. Numerics - Integral in Log-Log Space. `https://scicomp.stackexchange.com/questions/20901/integral-in-log-log-space`.

[4] Mathworks: Documentation. `https://www.mathworks.com/help/matlab/ref/logspace.html`.