Abstract  Spherical centroidal Voronoi tessellations (SCVT) are used in many applications in a variety of fields, one being climate modeling. They are a natural choice for spatial discretizations on the Earth, or any spherical surface. The climate modeling community, which has started to make use of SCVTs, is beginning to focus on exa-scale computing for large scale climate simulations. Due to this, a need is brought to light for fast and efficient grid generators. Current high resolution simulations on the earth call for a spatial resolution of about 11.1 km. In terms of a SCVT this corresponds to a quasi-uniform SCVT with roughly 2 million Voronoi cells. Computing this grid in serial is very expensive, and can take on the order of weeks to converge sufficiently for the needs of climate models. Utilizing conformal mapping techniques, as well as planar triangulation algorithms, and basic domain decomposition, this paper outlines a new algorithm that can be used to compute SCVTs in parallel, thus reducing the overall time to convergence. This reduces the actual time needed to create a grid on the Earth, as well as allows for new techniques to be explored when modeling the climate.

Goal of Research
We are attempting to improve the overall performance of spherical centroidal Voronoi tessellations (SCVT) generators, by speeding up the computation of spherical Delaunay triangulations. The computation of the spherical Delaunay triangulation is the most computationally demanding part of Lloyd’s algorithm, so we know algorithms for computing SCVTs. There are several algorithms available to compute spherical Delaunay triangulations, however they all scale poorly with point size. To alleviate this issue, we are developing a parallel algorithm to Lloyd’s algorithm which allows for parallel computations of spherical Delaunay triangulations.

Delaunay Triangulations
A Delaunay triangulation includes a set of Delaunay triangles of a point set. To be Delaunay a triangle must satisfy the following:

• Circumcircle must be empty
• At least 3 points lie on perimeter of circumcircle
• 4 or more points on perimeter means non-unique triangulation

Figure 1: Delaunay triangulation, with circumcircles overlaid

Voronoi Diagrams
A Voronoi diagram is the orthogonal dual mesh to a Delaunay triangulation. All Voronoi cells satisfy the Voronoi property given in equation 1.

\[ V_j \in \mathbb{F}, \ V_j = \{ x \in \mathbb{F} | |x - x_j| < |x - x_k| \} \text{ for } j = 1, \ldots, n \text{ and } j \neq k \]  

Equation 1

Figure 2 shows a Voronoi diagram overlaid on top of the Delaunay triangulation shown in figure 1.

Figure 2: Voronoi diagram with corresponding Delaunay triangulation

Centroidal Voronoi Tessellations
Typically, a Voronoi diagram is defined by a point set, known as generators. This Voronoi diagram is called a Centroidal Voronoi Tessellation when the set of generators of the Voronoi diagram is also the set of cell centers of masses, as defined in equation 2.

\[ x_i = \int_{V_i} \rho(x) \, dx / \int_{V_i} \rho(x) \, dx \]  

Equation 2

Where \( \rho(x) \) defines a density function, which can control the resolution of the grid.

Motivation for New Algorithm
As the SCVT increases it’s resolution, the triangulation (STRIPACK) cost becomes significantly more expensive than the integration step.

Stereographic Projection
Stereographic projections are conformal, meaning they preserve angles. In addition, stereographic projections have the unique property that they project circles and their interiors to circles and their interiors.

SCVT Based Domain Decomposition
In any parallel algorithm, some domain decomposition must be done to ensure load balancing and to help distribute the work to various processors. In this algorithm a coarse SCVT is used to define update boundaries, and various sorting algorithms can be used to determine which points each region is supposed to triangulate.

Non-Delaunay Triangles
A triangle that is non-Delaunay is easily defined as having the following criteria.

\[ \cos^{-1}(c_j, c_j + (s_j - 1)) > R \]  

These triangles need to be removed from each regions planar triangulation.

Parallel Lloyd’s Algorithm
Using coarse SCVT domain decomposition and stereographic projections, the parallel version of Lloyd’s algorithm is as follows.

• Define starting point set
• Start Iteration Loop
• Sort point into regions
• Triangulate points in planes
• Map triangulation back onto sphere
• Remove Non-Delaunay triangles
• Integrate Triangles to update point set
• Check for convergence
• Communicate updated point set
• If Converged, stop, otherwise loop again
• Final point set, and triangulation

Quasi-Uniform Results
Comparing serial STRIPACK to Parallel algorithm using Triangle for 163842 generators.

Parallel Performance Results

Future Work
• Variable Resolution Results
• Performance Optimization
• Better Sort Algorithm

References