Local Grid Refinement for a Nonlocal Model of Mechanics

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Models [1]

Linearized Peridynamics Model for Microelastic Materials

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_{\mathbf{x}}} c\frac{(\mathbf{x}'-\mathbf{x})\bigotimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t)$$

where $H_{\mathbf{x}}$ is a neighborhood of $\mathbf{x}$.

- Steady-state, one dimensional model with a "boundary" condition

$$\begin{cases} \frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x')-u(x)}{|x'-x|}dx' = b(x), & x\in\Omega \\ u(x) = g(x), & x\in\Gamma \end{cases}$$

$\Omega = (\alpha,\beta)$, $\Omega' = (\alpha-\delta,\beta+\delta)$, $\Gamma = \bar{\Omega}'\backslash\Omega = [\alpha-\delta,\alpha]\cup[\beta,\beta+\delta]$.

- Steady-state, two dimensional model with a "boundary" condition

$$\int_{H_x} c\begin{pmatrix} \frac{(x-x')^2}{[(x-x')^2+(y-y')^2]^{3/2}} & \frac{(x-x')(y-y')}{[(x-x')^2+(y-y')^2]^{3/2}} \\ \frac{(x-x')(y-y')}{[(x-x')^2+(y-y')^2]^{3/2}} & \frac{(y-y')^2}{[(x-x')^2+(y-y')^2]^{3/2}} \end{pmatrix}$$

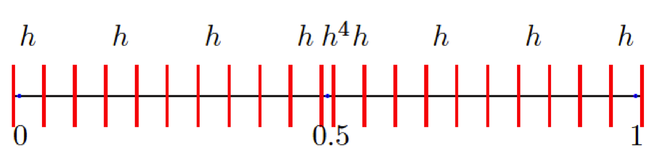
$$\begin{pmatrix} u_1(x,y)-u_1(x',y') \\ u_2(x,y)-u_2(x',y') \end{pmatrix} dx'dy' = \begin{pmatrix} b_1(x,y) \\ b_2(x,y) \end{pmatrix}, \quad x,y\in\Omega$$

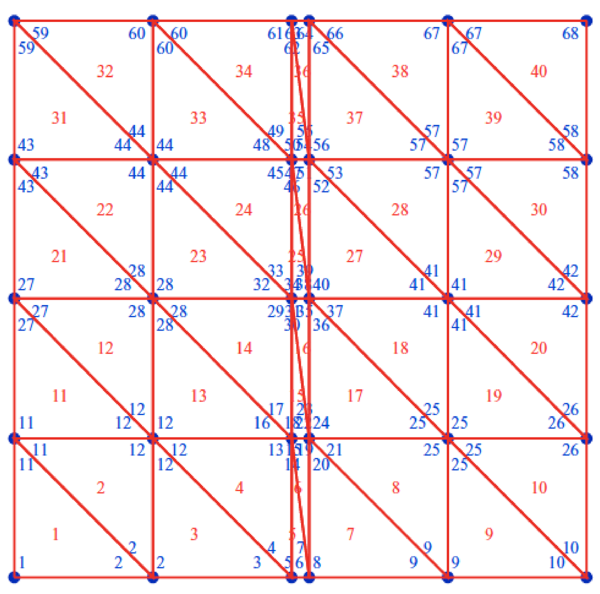
$$\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}, \quad x,y\in\Gamma$$

$\Omega = (\alpha_x,\beta_x)\times(\alpha_y,\beta_y)$, $\Omega' = (\alpha_x-\delta,\beta_x+\delta)\times(\alpha_y-\delta,\beta_y+\delta)$, $\Gamma = \bar{\Omega}'\backslash\Omega$.

Motivation from 1D and 2D Results [1, 2]

- Exact solution has a jump discontinuity at a point in 1D and along a straight line in 2D.
- Garlerkin Finite Element Method for Numerical Computation.
  - * CL — continuous piecewise linear functions.
    Pro: fewer degrees of freedom per triangle.
    Con: lower accuracy for discontinuous solutions.
  - * DL — discontinuous piecewise linear functions.
    Con: more degrees of freedom per triangle.
    Pro: higher accuracy can be achieved for discontinuous solutions.
- If a grid point is located at the point of discontinuity, $\|u-u^h\|_{L^2(\Omega')} = O(h^2)$, $\|u-u^h\|_{L^\infty(\Omega')} = O(h^2)$ for DL and $\|u-u^h\|_{L^2(\Omega')} = O(h^{1/2})$, $\|u-u^h\|_{L^\infty(\Omega')} = O(h^0)$ for CL.
- If no grid point is located at the point of discontinuity, $\|u-u^h\|_{L^2(\Omega')} = O(h^{1/2})$ and $\|u-u^h\|_{L^\infty(\Omega')} = O(h^0)$ for both DL and CL.
- However, if one does abrupt local refinement with an element of width h^4 surrounding the discontinuity, then for DL $\|u-u^h\|_{L^2(\Omega')} = O(h^2)$ and, if one excludes the elements containing the discontinuity, $\|u-u^h\|_{L^\infty(\Omega')} = O(h^2)$ as well.
- Combine the advantages of CL and DL method, we can do an abrupt local refinement $h \rightarrow h^4$, impose DL on the discontinous intervals/elements, and CL on other intervals/elements.





Problem

Abrupt refinement in 2D and 3D is difficult: the discontinuous path is usually unkown. Even if it is known, it is hard to refine if it has a more complicated shape such as a curved line. Adaptive refinement is needed.

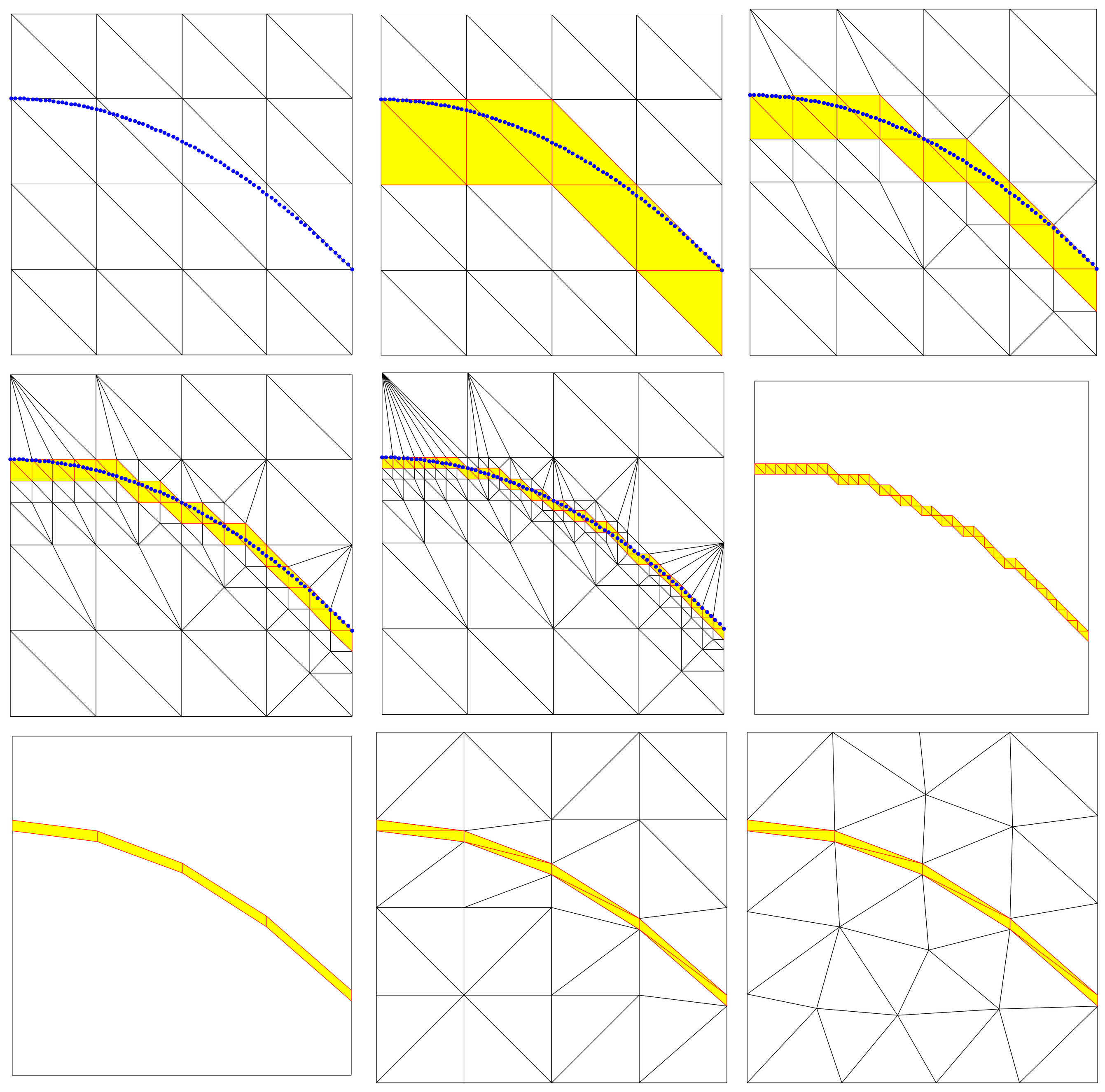
Local Grid Refinement

Goals of Refinement Strategy

1. Refine locally to get a thin layer of elements containing the discontinuity
   - the elements in the layer should have thickness $O(h^4)$ across the layer
   - but they should be $O(h)$ parallel to the layer
2. The elements outside the layer should not be thin and have linear dimension $O(h)$
3. The transition between the elements in the layer to those outside the layer should be abrupt
   - no transition zone from thin to regular elements

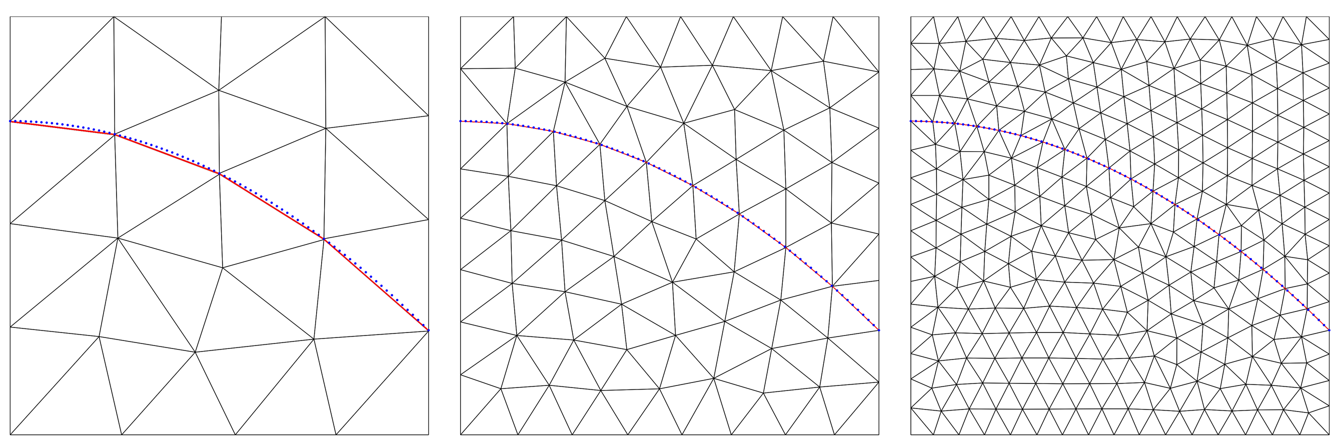
Steps of Refinement

- Step 1. Mark the triangles covering the discontinuous path yellow, split them into small identical triangles. Repeat this until small size achieved;
- Step 2. Polish the yellow triangles into long quadrilaterals which have width of $O(h^4)$ and length of $O(h)$;
- Step 3. Only retain the vetices of the yellow triangles and put the initial grid points back to the mesh. Use constrained centroidal Voronoi tessellation to rearrange the grid points in order to have triangles of better shapes.



Meshes with Different Grid Sizes

As the grid becomes finer, the computational complexity increases rapidly and thus divide and conquer method and parallel computing is applied to generate meshes in a reasonable time period.



Future Work

1. Solve 2D peridynamics equations using locally refined grids for cases where the position of the discontinuity is known;
2. Use adaptive strategies to identify the position of the discontinuity as one refines the grid;
3. Extend to 3D models.

References

[1] X. Chen, M. Gunzburger, *Continuous and discontinuous finite element methods for a peridynamics model of mechanics*, Computer Methods in Applied Mechanics and Engineering, Volume 200, 2011, 1237-1250.

[2] X. Chen, *Numerical Methods for Deterministic and Stochastic Nonlocal Problem in Diffusion and Mechanics*, Dissertation, 2012.

[3] Q. Du, M. Gunzburger, R. Lehoucq, Kun Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Review, Volume 54, Number 4, 667-696, 2012.

[4] Q. Du, M. Gunzburger, R. Lehoucq, K. Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Mathematical Modeling and Methods in Applied Science, Volume 23, 2013, 493-540.