A Multiscale Implementation for Nonlocal Model in 1D

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The Nonlocal Model

Nonlocal volume-constraint problem

- The *nonlocal volume-constrained problem* is given by

$$\begin{cases} -\mathcal{L}(u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega_{\mathcal{I}} \end{cases}$$

where $\mathcal{L}u = 2\int_{\mathbb{R}^n} (u(\mathbf{x}') - u(\mathbf{x}))\gamma(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$, γ denotes a symmetric kernel, i.e., $\gamma(\mathbf{x}, \mathbf{x}') = \gamma(\mathbf{x}', \mathbf{x})$ for all $\mathbf{x}, \mathbf{x}'$.

The kernel

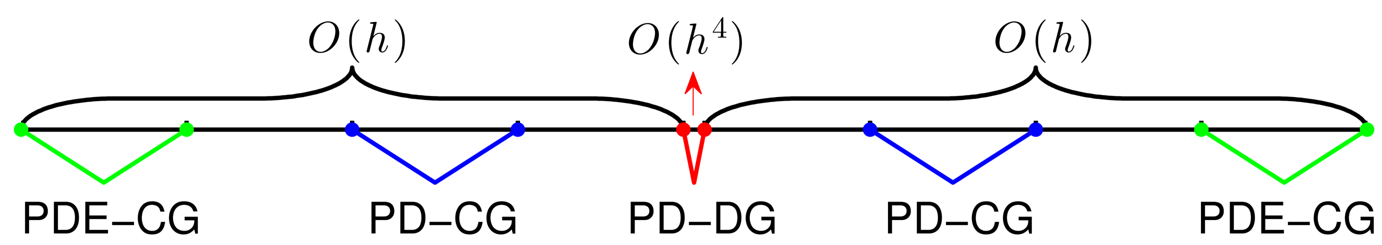
- We consider kernels of the form

$$\gamma(\mathbf{x}, \mathbf{x}') = \frac{1 - 1s}{\delta^{2-2s}} \frac{1}{|\mathbf{x} - \mathbf{x}'|^{n+2s}} \mathbb{1}_{H_{\mathbf{x}}}, \tag{1}$$

where n denotes the spatial dimension, s a constant, $\mathbb{1}$ the indicator function, $H_{\mathbf{x}} = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}'| \le \delta\}$, and $\delta > 0$ the horizon. If $s < 0$, the kernel (1) is integrable, i.e., we have that, for some constant $c^*(\delta) > 0$ whose value depends on δ,

$$\int_{\mathbb{R}^n} \gamma(\mathbf{x}, \mathbf{x}')d\mathbf{x}' \le c^*(\delta) < \infty \quad \forall \mathbf{x} \in \Omega. \tag{2}$$

A Multiscale Implementation



A multiscale implementation of the PD model would start with a choice for the bulk grid size h and a horizon parameter δ and then include the following components:

1. detection of elements that contain a discontinuity in the displacement;
2. refinement of the grid as necessary near the discontinuities;
3. use of DG for PD in regions containing the discontinuity;
4. use of CG for PD in regions neighboring the discontinuity;
5. use of CG for PDE if sufficiently far away from the discontinuity;
6. use of quadrature rules that can be applied for any combination of h and δ.

Posterior error estimator

For any element K in a mesh, the residual error is defined as

$$R^h(\mathbf{x}) = f(\mathbf{x}) - \mathcal{L}(u^h(\mathbf{x})) \quad \forall \mathbf{x} \in K.$$

If γ satisfies (1) with $s \in [0, 1)$, the posterior error is defined by

$$\widetilde{\eta}(u^h, K) = h^{2s}\|R^h\|_{L^2(K)}, \tag{3}$$

where s is defined in (1). For kernels satisfying (2), the posterior error estimator now takes the form

$$\widetilde{\eta}(u^h, K) = \frac{\|R^h\|_{L^2(K)}}{\sqrt{c^*(\delta)}},$$

where $c^*(\delta)$ is defined in (2). The total error estimator over $\Omega = (a, b)$ is then given by

$$\widetilde{\eta}(u^h, \Omega) = \Big(\sum_K \widetilde{\eta}^2(u^h, K)\Big)^{1/2}.$$

To be able to detect the elements containing points at which the solution is discontinuous, we also define the *grid-size weighted* posterior estimators

$$\eta^2(K) = \frac{\widetilde{\eta}^2(K)}{|K|} \quad \forall K \in \mathcal{T}_k \quad \text{and} \quad \eta^2(\Omega) = \sum_{K \in \mathcal{T}_k} \eta^2(K), \tag{4}$$

where $|K|$ denotes the length of the element K. Let $\widehat{K} \in \mathcal{T}_k$ denote an element that contains a point at which the solution has a jump discontinuity, we now have

$$\eta^2(\widehat{K}) = \frac{\widetilde{\eta}^2(\widehat{K})}{|\widehat{K}|} = \frac{O(h_{\widehat{K}})}{h_{\widehat{K}}} = \frac{O(h^4)}{h^4} = O(1) \quad \text{as } k \to \infty$$

and

$$\eta^2(K) = \frac{\widetilde{\eta}^2(K)}{|K|} = \frac{O(h^4)}{h} = O(h^3) \quad \text{as } k \to \infty,$$

Adaptive Refinement Algorithm

Algorithm 6.1 ()

Input: Dörfler marking parameter $\theta \in (0, 1]$, a grid size h,
a uniform triangulation $\mathcal{T}_0$ of Ω into elements of size h,
a set $\mathcal{P}_0$ of nodes consisting of $\mathcal{T}_0$.
Initialization: $k = 0$; **foreach** *element* K *in* $\mathcal{T}_0$ **do** set Group$(K) = 1$;
while *true* **do**
1. **foreach** *node* p *in* $\mathcal{P}_k$, *the set of nodes consisting of* $\mathcal{T}_k$ **do** if p is an endpoint of a Group 1 element K, set p as a PD-DG node;
   elseif p is an endpoint of an element within the δ-neighborhood of any Group 1 element, set p as PD-CG node;
   otherwise, set p as PDE-CG node;
2. using the triangulation $\mathcal{T}_k$, solve the multiscale problem for $u_k^h(x)$;
3. **foreach** *element* K *in* $\mathcal{T}_k$ **do** compute the error indicator $\eta(u_k^h, K)$;
4. define the set $\mathcal{M}_k$ of elements contributing the proportion θ of the total error, i.e., $\eta^2(u_k^h, \mathcal{M}_k) \ge \theta\eta^2(u_k^h, \Omega)$, with a minimal cardinality;
5. **foreach** *element* K *in* $\mathcal{T}_k$, **do** Group$(K) = 1$ if $K \in \mathcal{M}_k$, else *Group*$(K) = 2$;
6. create triangulation $\mathcal{T}_{k+1}$ of the Group 2 elements with coarsened $O(h)$ mesh;
7. **if** *all Group 1 elements have size* $O(h^4)$ **then** do steps 1, 2, and 3 again with the coarsened final grid and then break;
8. modify triangulation $\mathcal{T}_{k+1}$ by refining Group 1 elements;
9. set $k := k + 1$;

end

Numerical Results

The manufactured solution used is given by

$$u(x) = \begin{cases} x & \text{if } x < \widetilde{x} \\ x^2 & \text{otherwise,} \end{cases} \tag{5}$$

We set the discontinuity in the solution to be located at $\widetilde{x} = 0.503$ and apply Algorithm 6.1 with $\delta = 0.02$ and kernel $\gamma = \frac{3}{2\delta^3}\mathbb{1}_{H_{\mathbf{x}}}$.

The following tables present the numerical results for several errors, e.g., L^2 error, H^1 error, L^∞ error, energy error and posterior error $\widetilde{\eta}$. N_0^n is the initial number of nodes.

errors and convergence rates including all elements								
N_0^n	$\|e^h\|_{L^2}$	CR	$\|e^h\|_{L^\infty}$	CR	$\|\|\|e^h\|\|\|$	CR	$\widetilde{\eta}(\Omega)$	CR
5	8.74e-3	–	1.29e-1	–	5.59e-1	–	5.53e-1	–
9	2.19e-3	1.98	1.37e-1	-0.08	1.41e-1	1.97	1.37e-1	2.00
17	5.24e-4	2.06	1.35e-1	0.02	3.66e-2	1.95	3.47e-2	1.98
33	1.29e-4	2.01	1.35e-1	0.00	1.08e-2	1.76	1.12e-2	1.64
65	3.12e-5	2.05	1.38e-1	-0.03	2.49e-3	2.12	2.44e-3	2.19
129	7.87e-6	1.99	1.37e-1	0.01	6.32e-4	1.98	6.35e-4	1.94
257	1.92e-6	2.05	1.64e-1	-0.26	1.56e-4	2.03	1.55e-4	2.04

errors excluding the element containing the discontinuity						
N_0^n	$\|e^h\|_{L^2}$	CR	$\|e^h\|_{L^\infty}$	CR	$\|e^h\|_{H^1}$	CR
5	7.49e-3	–	1.50e-2	–	1.01e-1	–
9	1.89e-3	1.97	3.82e-3	1.96	5.06e-2	0.99
17	4.58e-4	2.05	9.57e-4	2.00	2.53e-2	1.00
33	1.09e-4	2.07	2.39e-4	2.00	1.26e-2	1.00
65	2.66e-5	2.04	6.00e-5	2.00	6.32e-3	1.00
129	6.75e-6	1.98	1.50e-5	2.00	3.16e-3	1.00
257	1.64e-6	2.06	3.76e-6	2.01	1.58e-3	1.01

The figures below show the plots of the numerical solutions at the initial and final step of the adaptive algorithm for $h = 1/8$.