On the Spatial and Temporal Order of Convergence of Hyperbolic PDEs

Siddhartha Bishnu[1,2], Mark Petersen[1] and Bryan Quaife[2]

[1]Computer, Computational and Statistical Sciences Division, Los Alamos National Laboratory
[2]Scientific Computing Department, Florida State University

Motivation

- A graduate level textbook on numerical analysis typically contains predictor-corrector and multistep time-stepping methods for advancing ODEs in the first few chapters, followed by spatial discretization operators of PDEs in separate ones.
- In real world applications, the discretization of a hyperbolic PDE consists of both spatial and temporal components, and the order of convergence of a hyperbolic PDE with spatial and/or temporal refinement is a function of both the mesh spacing Δx and the time step Δt.
- I investigate the simultaneous dependence of the local truncation error of the numerical solution of a hyperbolic PDE on Δx and Δt, for varying orders of spatial and temporal discretizations.

Truncation Error of a Hyperbolic PDE

Theorem 1. Assuming the existence of a smooth solution of a hyperbolic PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$ at spatial locations x_j for $j = 1, 2, \cdots$ at time level t^n, the exact solution at spatial location x_j and time level $t^{n+1} = t^n + \Delta t$ is

$$u_j^{n+1} = u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\frac{\partial^k u}{\partial t^k}\right)_j^n \equiv u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\mathcal{F}^{(k)}\right)_j^n,$$

and the numerical solution at spatial location x_j and time level t^{n+1} obtained with a time-stepping method of order β, belonging to the Method of Lines, is

$$\hat{u}_j^{n+1} = u_j^n + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left(\hat{\mathcal{F}}^{(k)} + O\left(\Delta x^{\alpha}\right)\right)_j^n,$$

where $\left(\hat{\mathcal{F}}^{(k)}\right)_j^n = \left(\mathcal{F}^{(k)}\right)_j^n \equiv \left(\frac{\partial^k u}{\partial t^k}\right)_j^n$ for $k = 1, 2, \cdots, \beta$ and α represents the order of spatial discretization. The local truncation error can be expressed as

$$\begin{aligned}
\hat{\tau}_j^{n+1} &= u_j^{n+1} - \hat{u}_j^{n+1} \\
&= \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \left\{\left(\mathcal{F}^{(k)}\right)_j^n - \left(\hat{\mathcal{F}}^{(k)}\right)_j^n + O\left(\Delta x^{\alpha}\right)\right\} \\
&= \sum_{k=1}^{\beta} \frac{\Delta t^k}{k!} O\left(\Delta x^{\alpha}\right) + \sum_{k=\beta+1}^{\infty} \frac{\Delta t^k}{k!} \left\{\left(\mathcal{F}^{(k)}\right)_j^n - \left(\hat{\mathcal{F}}^{(k)}\right)_j^n + O\left(\Delta x^{\alpha}\right)\right\} \\
&= \frac{\Delta t}{1!} O\left(\Delta x^{\alpha}\right) + \frac{\Delta t^2}{2!} O\left(\Delta x^{\alpha}\right) + \cdots + \frac{\Delta t^{\beta}}{\beta!} O\left(\Delta x^{\alpha}\right) \\
&\quad + \frac{\Delta t^{\beta+1}}{(\beta+1)!} \left(c_{\beta+1} + O\left(\Delta x^{\alpha}\right)\right)_j^n + O\left(\Delta t^{\beta+2}\right) \\
&= \Delta t O\left(\Delta x^{\alpha}\right) + \Delta t^2 O\left(\Delta x^{\alpha}\right) + \cdots + \Delta t^{\beta} O\left(\Delta x^{\alpha}\right) + O\left(\Delta t^{\beta+1}\right),
\end{aligned}$$

where $\left(c_{\beta+1}\right)_j^n = \left(\mathcal{F}^{(\beta+1)}\right)_j^n - \left(\hat{\mathcal{F}}^{(\beta+1)}\right)_j^n \neq 0$. The global truncation error at a time horizon is

$$\begin{aligned}
\left(\hat{\tau}_G\right)_j^{n+1} &= O\left(\Delta x^{\alpha}\right) + \Delta t O\left(\Delta x^{\alpha}\right) + \Delta t^2 O\left(\Delta x^{\alpha}\right) + \cdots + \Delta t^{\beta-1} O\left(\Delta x^{\alpha}\right) + O\left(\Delta t^{\beta}\right) \\
&\approx O\left(\Delta x^{\alpha}\right) + O\left(\Delta t^{\beta}\right) \quad \text{for } \Delta t \ll 1.
\end{aligned}$$

Convergence of Global Solution Error

- Consider a stable numerical scheme with spatial discretization $\sim O\left(\Delta x^{\alpha}\right)$ and a time-stepping method $\sim O\left(\Delta t^{\beta}\right)$, and assume that the global solution error is of the same order of accuracy as the global truncation error.
- For modeling the hyperbolic PDE $u_t = \mathcal{F}\left(u, u_x, u_{xx}, \cdots, x, t\right)$, the following hold in the asymptotic regime, where the truncation error is dominated by the powers of Δx and Δt rather than their coefficients:

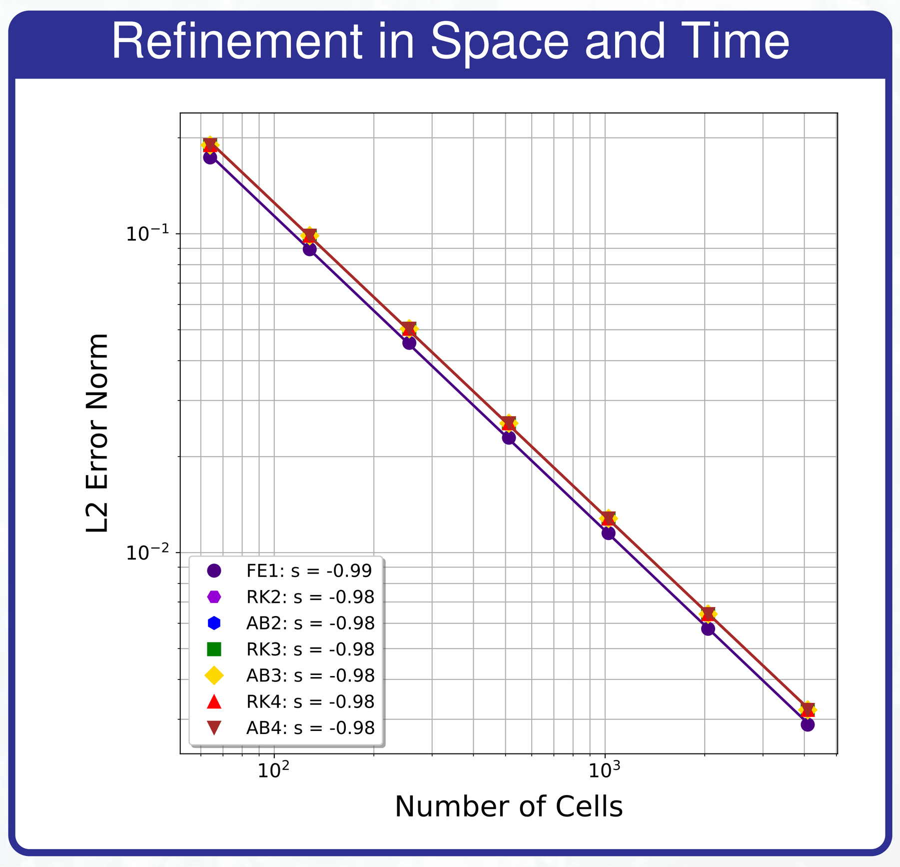
Refinement in Space: $\Delta x \to 0$, Δt fixed	Refinement in Time: $\Delta t \to 0$, Δx fixed	Refinement in Space and Time: $\Delta x \to 0$, $\Delta t \to 0$, $\Delta t/\Delta x$ fixed
Convergence Not Attained: Why? $O\left(\Delta t^{\beta}\right)$ Dominates	Convergence Not Attained: Why? $O\left(\Delta x^{\alpha}\right)$ Dominates	Convergence Attained at Order of Minimum of α and β in the Asymptotic Regime

- A simultaneous refinement of Δt and Δx, at $\Delta t/\Delta x = \gamma$, a constant, yields
$$\hat{\tau}_G \approx O\left(\Delta x^{\alpha}\right) + O\left(\Delta t^{\beta}\right) = O\left(\Delta x^{\alpha}\right) + O\left(\gamma^{\beta} \Delta x^{\beta}\right) \approx O\left(\Delta x^{\min(\alpha,\beta)}\right).$$
- **Strategy:** Given α, we need $\beta \geq \alpha$ to obtain maximum possible order of accuracy. But we gain no improvement in order of convergence for $\beta > \alpha$ despite more work. So, optimum choice is $\beta = \alpha$.
- **Refinement Only in Space or Only in Time:** By comparing the numerical solution or the error between two successive spatial (or temporal) resolutions, one can verify the order of the spatial (or temporal) discretization.

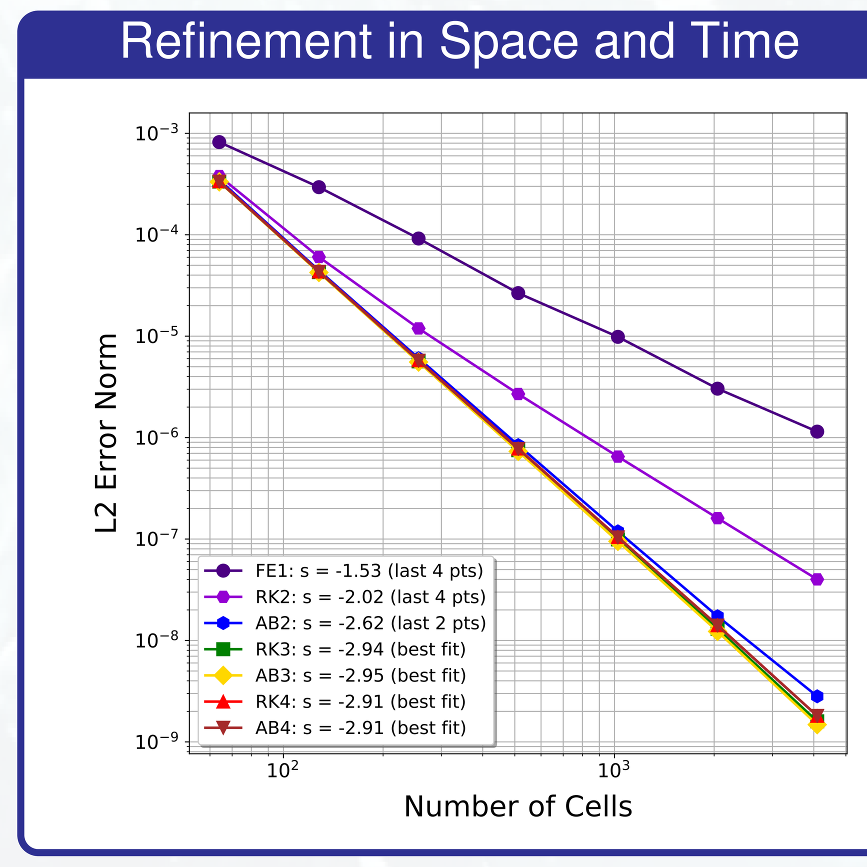
Numerical Results

- We perform convergence studies with a linear variable-coefficient advection equation discretized in space with first-order upwind finite difference scheme and piecewise parabolic reconstruction (PPR), and advanced in time with first-order Forward Euler (FE1), and Runge-Kutta (RK) and Adams-Bashforth (AB) methods, from second to fourth order.
- Starting with the mean solution in each cell, PPR interpolates the solution to the edges. This interpolation is fourth-order accurate on a uniform mesh. Then it applies the monotonized-central slope limiter, and adjusts the edge estimates to flatten any local maximum or minimum, to ensure monotonicity.
- The following figures show convergence plots using first-order upwind (first column) and PPR (second column) in space, and the above-mentioned time-stepping methods: refinement in both space and time (first row), refinement only in space (second row), and refinement only in time (third row).

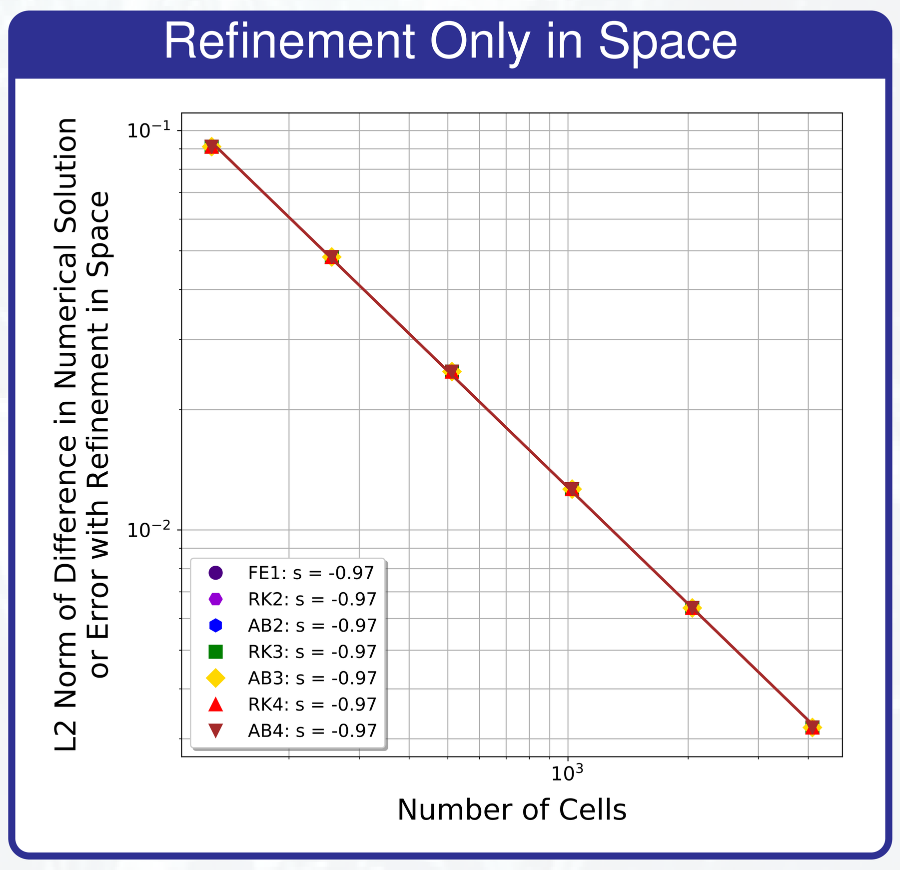
Refinement in Space and Time



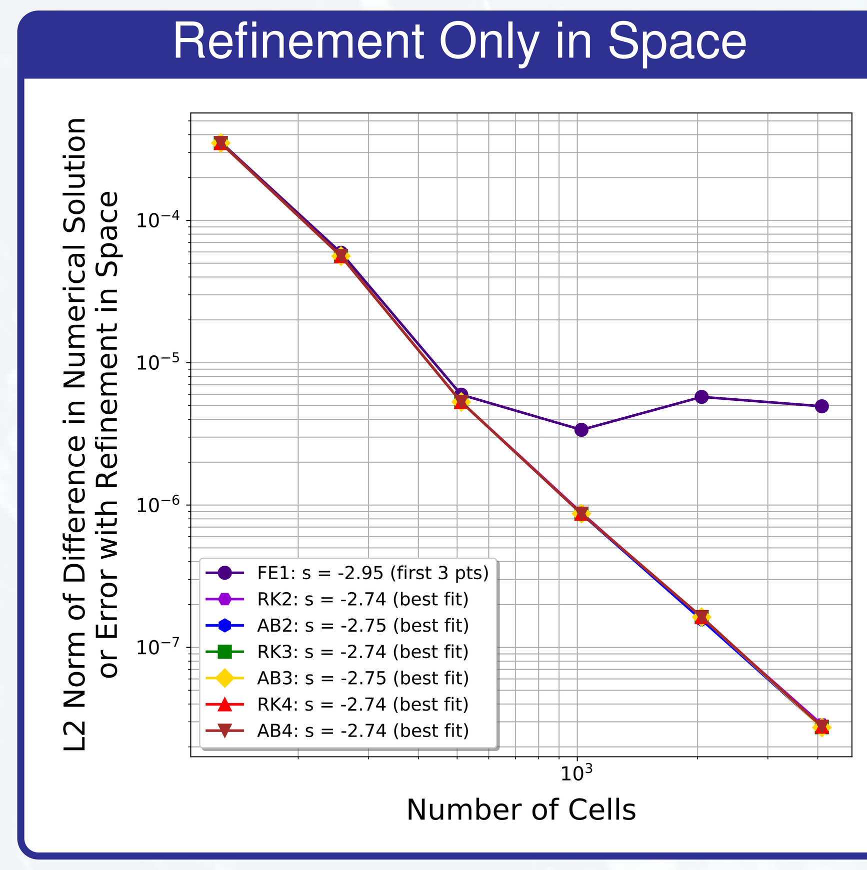
Refinement in Space and Time



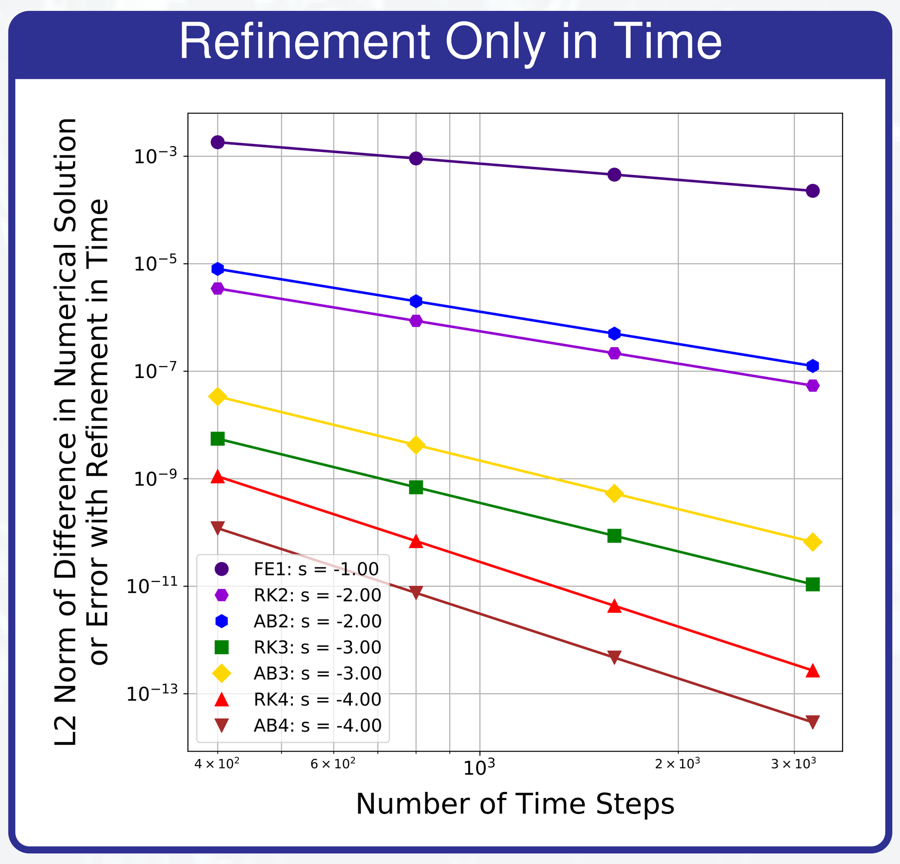
Refinement Only in Space



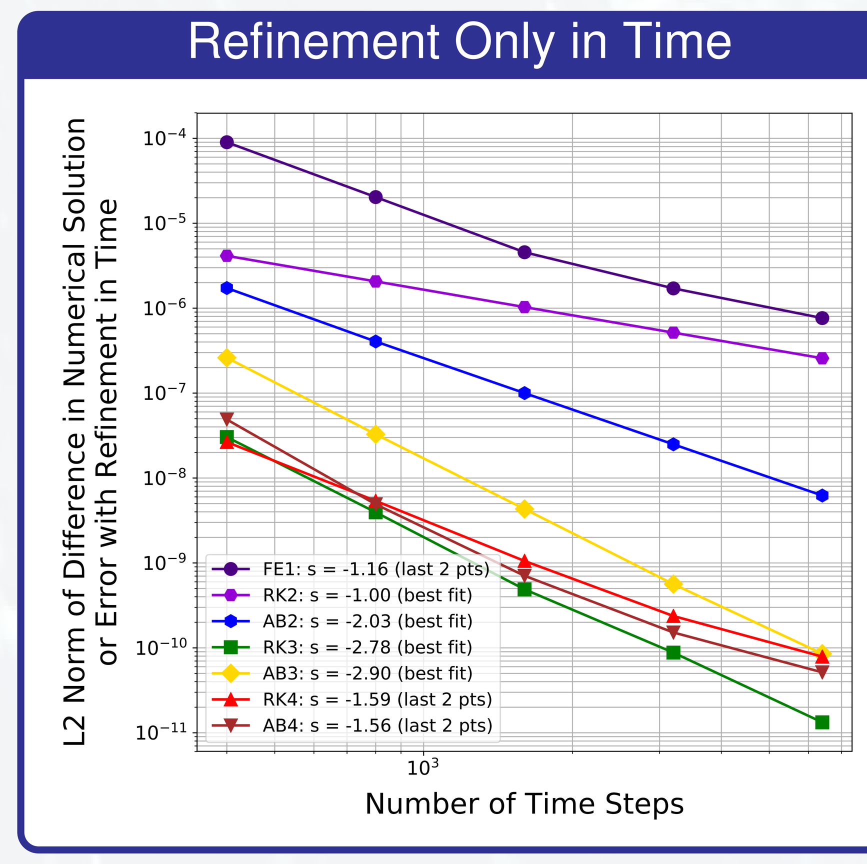
Refinement Only in Space



Refinement Only in Time



Refinement Only in Time



Conclusion and Future Work

- With the finite difference method, we have already reached the asymptotic regime, and the order of convergence is exactly what is predicted by our theory.
- With the finite volume method using PPR in space, we have not yet reached the asymptotic regime for some time-stepping methods, and may not be able to do so before machine precision error dominates. However, we are approaching the asymptotic regime, as evidenced by the reduction in the order of the convergence slopes.
- The slope-limiting monotonicity-preserving strategies of PPR can reduce the spatial and temporal orders of accuracy.
- Ongoing and future work entails extending our theory to parabolic PDEs and high order discretizations in space and time.